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DESIGN METHODS OF WAVELET AND MULTIWAVELET FILTER BANKS

ABSTRACT

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PREFACE

Problem statement

In recent decades, scalar wavelets have established themselves as an indispensable tool in signal processing, in applications such as numerical analysis, operator theory, denoising and compression of *N*-D signals, object extraction from astronomical images, machine learning, data sorting, database searching, time series analysis, computational medicine, and others. They have been around since the late 1980s. There are different types of them: wavelet packets, ridgelets, curvelets, slantlets, frames and other constructions. Since the early 1990's, one known generalization of scalar wavelets have been multiwavelets. The great interest in them is caused by the fact that they contain more than one function while possessing at the same time the most important characteristics from the theory of filter banks - short support, symmetry, and vanishing moments of high degree.

An important subfield is orthogonal multiwavelet filters, the construction of which requires satisfying a number of restrictive conditions. They have advantageous properties, but are very difficult to design. Generally, new design methods and algorithms are necessary.

Motivation

There is a significant difficulty and a major challenge in finding orthogonal multiscaling functions by spectral decomposition of a singular matrix filter product. Moreover, due to the presence of single or multiple zeros in the determinant of the product filter, the spectral decomposition may be highly erroneous or even impossible.

The main research question is the extension of wavelet and multiwavelet theory. This research contributes four methods for constructing scalar or vector scaling functions from different polynomials and splines:

- *Basis change method* ;
- *Brute force method;*
- *Inner product method;*
- *Bauer's method for spectral factorization.*

The first three methods are based on polynomial basis functions or different splines. These can be exponential functions, Legendre polynomials in [-1,1] or [0,1], cubic or quintic Hermitian polynomials in [0,1] and [0,2], Chebyshev polynomials of the first and second kind, Haar and Walsh functions, etc.

The fourth proposed method constructs scaling and multiscaling functions satisfying desired properties - orthogonality, vanishing moment, compact support, and smoothness. Two numerical algoritms for the Bauer's method for spectral factorization of scalar and matrix product filters are construted.

Main tasks of the scientific research:

1. To conduct a study, overview and critical analysis of existing methods for constructing scaling and multiscaling functions;

2. To outline research opportunities in constructing new methods of implementing filter banks;

3. To propose new methods for constructing scaling and multiscaling functions from polynomials and splines;

4. To define and construct spectral factorization methods;

5. To elaborate on the algorithms for a fast and accurate Bauer's method;

6. To implement an Alpert multiwavelet filter bank based on the lifting scheme and perform experimental studies of the methods for constructing wavelet and multiwavelet filter banks.

Content structure

The dissertation consists of 5 chapters, preface, conclusion, and references.

The *Preface* outlines the topic, object and subject of the dissertation work, as well as the leading hypotheses. The problem statement and the motivation for conducting the dissertation research are briefly described. The purpose and methodology of the research work, as well as the methods according to which it is to be achieved, are given.

Chapter 1 is devoted to existing methods for constructing wavelet and multiwavelet filter banks. It includes the theory of basis functions from polynomials and splines, spectral decomposition, and construction of scalar and vector filter banks.

Chapter 2 presents three construction methods for scaling and multiscaling functions from polynomials and splines.

Chapter 3 is devoted to existing methods for spectral factorization. The necessary condition for smoothness of the matrix product filter is described. The *Daubechies 4* scaling function is designed according to the root method for spectral factorization. Bauer's theoretical method for spectral factorization is also described and an Alpert product filter is obtained.

Chapter 4 is devoted to the development of fast algorithms implementing Bauer's method and their numerical solution using three numerical methods. Algorithms 1 and 2 for a fast and accurate Bauer method for scalar and vector spectral decomposition are developed and Alpert orthogonal multifilter banks are constructed.

Chapter 5 is devoted to the following novel aspects:

1) Comparative analysis of the four methods for constructing scaling and multiscaling functions, Bauer methods for spectral decomposition for Haar and Daubechies scaling functions, as well as Alpert multiscaling function;
2) Empirical research on the fast and exact Bauer's method for scalar and matrix spectral factorization – Haar and Daubechies 4 scaling functions;
3) Seven examples of *Algorithm 1, 2* implemeting the exact Bauer's method;
4) Built–in functions for the seven examples of *Algorithm 1, 2*;
5) The lifting scheme for the Alpert multiwavelet filter bank is implemented. It is applied with different quantizations: $\sqrt{3}$ for 256 ×256 and 512 ×512 image denoising with gray levels and AWGN with (σ = 10, 20);
6) Image compression of astronomy images from scanned photograph plates are compared.

CHAPTER 1
REVIEW OF CONSTRUCTION METHODS OF WAVELET AND MULTIWAVELET FILTER BANSKS

1.1 Introduction

Wavelets are, essesntially, short or fast-decaying waves where by translation and dilation one can obtain a set of functions possessing the important properties of orthogonality, zero moments, compact support, and smoothness. They are particularly useful in the analysis of non-stationary signals, which requires that the frequency characteristics of a given filter depend on time. They are obtained with the construction of filter banks where spatial and frequency features are simultaneously determined for a given signal, which is impossible with Fourier transformation.

In wavelet theory, there are spline wavelets obtained from different spline functions. They are a linear combination of B-splines inheriting their basis functions and a compact support but the resulting functions are non-orthogonal. Because of their simple structure, they are one of the most important and interesting wavelet family. They are used in the construction of wavelet finite elements that satisfy the condition of continuity of a shape function. In this way, multi-level (multi-scale) representation is achieved in many engineering problems. This achieves the desired accuracy and provides adaptive hierarchical solutions.

An important drawback of wavelets is the impossibility of simultaneously possessing all important properties - orthogonality, zero moments, compact support, and smoothness [131], [132]. To overcome this drawback, the multiscaling and multiwavelet functions have been used. This changes the structure of a filter bank where instead of one scaling and wavelet function, two or more functions called multiscaling and multiwavelet functions are used simultaneously.

The advantage of the multifilter theory is the possession of the above properties, which ensures fast signal recovery (at the expense of orthogonality), good efficiency (at the expense of smoothness), and a high degree of approximation (at the expense of a large number of zero moments). The symmetry of the function allows a symmetrical extension of the boundaries of signals. Orthogonality results in independent subimages. The higher degree of vanishing moments leads to the ability to represent polynomials of a higher degree with a smaller number of coefficients.

1.2. Brief theory of spline and Legendre polynomials

1.2.1 Spline functions

From a mathematical point of view, a '*spline*' is a partially linear function built from polynomial functions whose smoothness depends its derivatives. More generally, a spline is the set of all functions that are parts of a polynomial necessary to construct a function in the interval [a, b] with certain smoothness conditions. Splines whose polynomials are of low degree are called piecewise linear and are used as interpolating functions.

1.2.1.1 Linear B – spline ("*Hat*" function)

The '*Hat*" function is piecewise linear function, known as linera B – spline:

$$L(t) = at + b\,. \tag{1.1}$$

where a и b are coeffiecients. It can be used as a linear interpolant and/or finite element. Its multiscaling fucntion consists from three coefficients h_0, h_1, h_2:

$$\phi(t) = h_0\phi(2t) + h_1\phi(2t-1) + h_2\phi(2t-2) \quad . \tag{1.2}$$

1.2.1.2 Cubic Hermitain spline

Hermitian spline functions belong to a class of splines that are defective[(A)], i.e. piecewise polynomial function with constant defect on the knots. The most commonly used is the cubic spline consisting of a polynomial of degree ≤ 3 with two continuous derivatives for each subinterval. Cubic Hermitan spline is use for interpolation of data with continuous first and (possibly discontinuous) second derivatives of the nodes. Cubic Hermitan spline are constructed for two endpoints of a polynomial of the third degree $H(t) = at^3 + bt^2 + ct + d$ and its the derivatives (or tangents) at these points.

1.2.2 Legendre polynomials

Legendre polynomials are *t*-order polynomials, i.e. $1, t, \cdots, t^i$ and in [-1,1] forms nonorthogonal baisis:

$$\phi_r(t) = \sqrt{r + \frac{1}{2}} P_r(t) . \tag{1.3}$$

where:

$$\begin{aligned} &P_0(t) = 1, \\ &P_1(t) = t, \\ &P_2(t) = \frac{1}{2}(3t^2 - 1) \\ &(m+1)P_{m+1}(t) = (2m+1)tP_m(t) - mP_{m-1}(t) \end{aligned} \quad \text{, for } m = 1,2,3,\cdots \tag{1.4}$$

In the multiwavelet theory Legendre polynomials are in a interval [0,1]. The first multiscaling function is of Alpert [6] which is construted from the two functions $P_0(t)$ и $P_1(t)$:

$$\Phi(t) = \begin{bmatrix} \varphi_0 \\ \varphi_1 \end{bmatrix} = \left[\mathrm{C}_0 + \mathrm{C}_1 z^{-1} \right] = \begin{bmatrix} P_0(t) \\ P_1(t) \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3}(2t-1) \end{bmatrix}, \tag{1.5}$$

where C_0 and C_1 are matrix coefficients.

1.3. General theory of filter banks

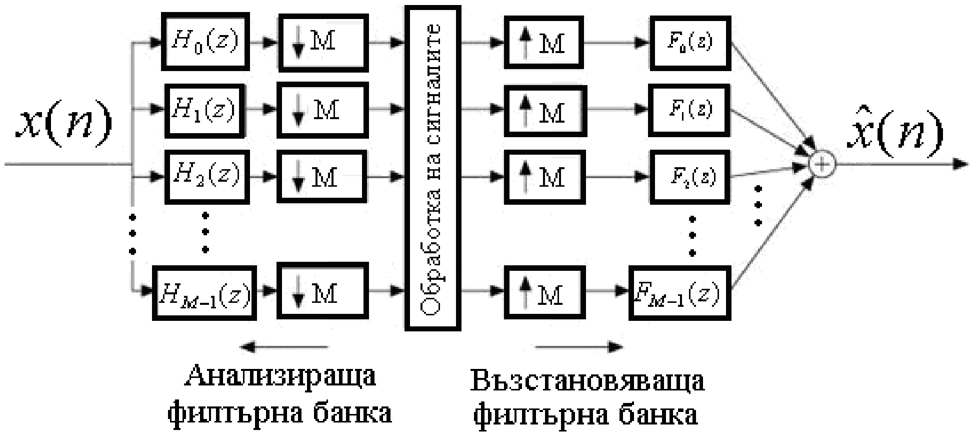
1.3.1 Scalar filter banks

1.3.1.1 General theory

Wavelet theory is based on a domain of basis functions interconnected by scaling and translation. At the heart of this domain is a wavelet basis function used to generate all other basis functions. It has certain characteristics in the L^2 space and is called the mother wavelet, wavelet function, or just wavelet, and is denoted by ψ(t). There is a second function in the basis, allowing it to be built from a finite number of functions. This function is the parent wavelet, a scaling function, and is denoted by ϕ(t). The integer translations of the two functions form a Riez basis.

(A) The difference between the degree of a spline and its smoothness is called the *spline defect*. For example, a piecewise linear continuous function (its graph is a polygonal line) is a spline of degree one of defect 1.

The two functions are obtained using a filter bank, which usually has a tree-like hierarchy of two different types - for analysis and recovery (synthesis) of functions. The filter bank is a combination of shift-dependent filters with coefficient M – downsampling, denoted with (↓M), and decimation (upsampling), denoted with (↑M). The structure of an r-channel filter bank is shown in fig. 1.4. This way of analyzing a function (signal) is called multi-resolution analysis (MRA). If the analysis and recovery regions consist of the same scaling and wavelet functions, the filterbank is called orthogonal, and when they consist of different functions, the filter bank is called biorthogonal.



Фиг.1.4. *M*–channels scalar filter bank

The two – channels filter bank with input signal *x*(n) is described by the equation:

$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}\left[H_0(z)F_0(z)+H_1(z)F_1(z)\right]\,X(z)+\frac{1}{2}\left[H_0(-z)F_0(z)+H_1(-z)F_1(z)\right]\,X(-z)\\ &= T(z)X(z)+S(z)X(-z)\end{aligned} \quad (1.6)$$

which is equivalent of the matrix product:

$$\hat{X}(z)=\left[F_0(z)\quad F_1(z)\right]\frac{1}{2}\begin{bmatrix}H_0(z) & H_0(-z)\\ H_1(z) & H_1(-z)\end{bmatrix}\begin{bmatrix}X(z)\\ X(-z)\end{bmatrix}.$$

The matrix product $H_0(z)F_0(z)$ is called a product filter. It is the foundation for finding of scaling and multiscaling functions by spectral factorization.

1.3.1.2 Scaling and wavelet functions by a parameter

The construction of orthogonal scaling and wavelet functions can be done by the poyphase matrix with a parameter [86],[129],[139] which is called a lattice structure:

$$H_p(z)=\left[\prod_{i=0}^{L-1}\begin{pmatrix}\cos\phi_i & \sin\phi_i\\ -\sin\phi_i & \cos\phi_i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix}\right]\begin{pmatrix}\cos\phi_L & \sin\phi_L\\ -\sin\phi_L & \cos\phi_L\end{pmatrix} \quad (1.7)$$

where $\phi_i\in[0,2\pi]$ and $\phi_L[0,\pi]$.

In other research [86] a parameter structure is used:

$$\begin{bmatrix} c_0^K & d_0^K \\ c_1^K & d_1^K \\ c_2^K & d_2^K \\ & \\ c_{N-1}^K & d_{N-1}^K \end{bmatrix} = \begin{bmatrix} \cos\phi_K & & \\ \cos\phi_K & & \\ & I \otimes R(\phi_K) & \\ & & \cos\phi_K \\ & & \cos\phi_K \end{bmatrix} \cdots \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{bmatrix} \text{за } K \geq 2 \tag{1.8}$$

where I is an identity matrix. In this way filter banks can be implemented directly.

Additionally, a universal parameterization is given [87]:

$$R(\phi) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = J(\alpha)J(\beta), \tag{1.9}$$

where $J(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$ is a Jacobi matrix with $\beta - \alpha = \phi \bmod 2\pi$. Then Jacobi matrix has three diagonal matrices:

$$\begin{aligned} R(\phi) &= I - A\left(\frac{\phi}{2}\right)B\left(\frac{\phi}{2}\right)A^T\left(\frac{\phi}{2}\right) = \\ &= I - \sqrt{2}\begin{bmatrix} 0 & \sin\frac{\phi}{2} \\ 1 & -\cos\frac{\phi}{2} \end{bmatrix}\begin{bmatrix} 1 & 2\cos\frac{\phi}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \sin\frac{\phi}{2} & -\cos\frac{\phi}{2} \end{bmatrix} \end{aligned} \tag{1.10}$$

1.3.1.3 Properties of scaling and wavelet functions

The subchapter describes more important properties of scaling and wavelet functions - Heisenberg's uncertainty principle, symmetry, flatness, and v*anishing moments*.

1.3.1.4 Multiresolution Analysis (MRA) [139]

A key concept in wavelet theory is the nested structure called multiresolution analysis (MRA) or multiscale approximation (MSA), consisting of successively coarser or finer spaces $V_{j,}$:

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots, \in L^2(R) \text{ (Lebesgue space)}$$

$\leftarrow$ *rougher space* *finer space* $\rightarrow$

MRA possesses following properties:

1. *Scaling* – For each *j*,

$$\text{a function } f(t) \in \mathrm{V_j} \text{ iff } f(2t) \in \mathrm{V_{j+1}};$$

2. *Inclusion* - For each *j*

$$\mathrm{V_j} \subseteq \mathrm{V_{j+1}};$$

3. *Completeness* - The union of all subspaces $\mathrm{V_j}$'s in subspace L^2 is dense:

$$\text{closure}\left\{\bigcup_{j\in Z} \mathrm{V_j}\right\} = L^2(R);$$

4. Uniqueness – The subspaces $\mathrm{V_j}$ have no intersection:

$$\left\{\bigcap_{j\in Z} \mathrm{V_j}\right\} = \{0\}. \qquad \#$$

Consequently, the j-level MRA applied on a function $f(t)$ is obtaned using scaling and wavelet coefficients according to:

$$f(t)=\sum_{n=0}^{N-1}h(n)\phi(t-n)+\sum_{j=1}^{J}\sum_{n=0}^{N-1}g_{2^j+n}(n)\psi(2^j t-n)\,. \tag{1.11}$$

1.3.2 Vector filter banks (Multifilters)

1.3.2.1 General theory

The vector filter banks (or multifilter) are a generalization of scalar wavelet filter bank theory. Essentially, they are MIMO systems consisting of r –scaling functions $\Phi=[\phi_0, \phi_1, \ldots, \phi_r]^T$ and r – wavelet functions $\Psi=[\psi_0, \psi_1, \ldots, \psi_r]^T$. This means that multiwavelet filter bank is consists of four analysis and synthesis multifilters which possess simultaneously the properties of orthogonality, symmetry, compact support, and vanishing moments.

A important difference between the scalar and vector filter banks is the number of subbands obtained in the decomposition. For example, an image decomposed at one level by a two-channel scalar filter bank forms four subbands (subimages) shown in fig. 1.11(a), while a two-channel vector filter bank forms sixteen subbands (subimages) shown in fig. 1.11(b).

(a)

HH	HG
GH	GG

(b)

H_0H_0	H_0H_1	H_0G_0	H_0G_1
H_1H_0	H_1H_1	H_1G_0	H_1G_1
G_0H_0	G_0H_1	G_0G_0	G_0G_1
G_1H_0	G_1H_1	G_1G_0	G_1G_1

Фиг. 1.11 A one level decomposition of a image; (a) scalar two channel filter bank (4 subbands); (b) two channel multifilter bank (16 subbands)

The input – output equation of the two channel vector filter bank is:

$$\begin{aligned}\hat{X}(z)&=\frac{1}{2}\left[G_0(z)H_0(z)+G_1(z)H_1(z)\right]X(z)\\&\quad+\frac{1}{2}\left[G_0(z)H_0(-z)+G_1(z)H_1(-z)\right]X(-z)\end{aligned} \tag{1.12}$$

or by the modulation matrix

$$\begin{aligned}\begin{bmatrix}\hat{X}(z)\\ \hat{X}(z)\end{bmatrix}&=\frac{1}{2}\begin{bmatrix}G_0(z) & G_1(z)\\ G_0(-z) & G_1(-z)\end{bmatrix}\begin{bmatrix}H_0(z) & H_0(-z)\\ H_1(z) & H_1(-z)\end{bmatrix}\begin{bmatrix}X(z)\\ X(-z)\end{bmatrix},\\&=G^m(z)H^m(z)\begin{bmatrix}X(z)\\ X(-z)\end{bmatrix}\end{aligned} \tag{1.13}$$

where, $X(z)$ is input vector signal,

$H_r(z)$ - analysis multifilters, $r=0,1$

$G_r(z)$ - synthesis multifilters, $r=0,1$

$\hat{X}(z)$ - output vector signal.

$H^{\mathrm{m}}(\mathrm{z})$ - orthogonal analysis modulation matrix,

$G^{\mathrm{m}}(\mathrm{z})$ - orthogonal synthesis modulation matrix.

The perfect reconstruction conditions of multiwavelet filter banks using the modulation matrices are:

$$\begin{aligned} \widetilde{H}^{\mathrm{m}}(\mathrm{z})H^{\mathrm{m}}(\mathrm{z}) = H^{\mathrm{m}}(\mathrm{z})\widetilde{H}^{\mathrm{m}}(\mathrm{z}) = cI \\ \widetilde{G}^{\mathrm{m}}(\mathrm{z})G^{\mathrm{m}}(\mathrm{z}) = G^{\mathrm{m}}(\mathrm{z})\widetilde{G}^{\mathrm{m}}(\mathrm{z}) = cI \end{aligned}, \tag{1.14}$$

where c is a constant, I – an identity matrix, and ($\tilde{\cdot}$) denotes the Hermitian matrix. Therefore, the conditions to obtain orthogonal multifilter banks are:

$$\begin{aligned} H_0(\mathrm{z})\widetilde{H}_0(\mathrm{z}) + H_0(-\mathrm{z})\widetilde{H}_0(-\mathrm{z}) = cI \\ H_1(\mathrm{z})\widetilde{H}_1(\mathrm{z}) + H_1(-\mathrm{z})\widetilde{H}_1(-\mathrm{z}) = cI \\ H_0(\mathrm{z})\widetilde{H}_1(\mathrm{z}) + H_0(-\mathrm{z})\widetilde{H}_1(-\mathrm{z}) = 0 \\ H_1(\mathrm{z})\widetilde{H}_0(\mathrm{z}) + H_1(-\mathrm{z})\widetilde{H}_0(-\mathrm{z}) = 0 \end{aligned} \tag{1.15}$$

which means that only the lowpass multifilter $H_0(\mathrm{z})$ is needed.

1.3.2.2 Properties of the multiscaling function

In this sub-chapter we consider some important properties:

A. *polynomial reproduction of discrete polynomials;*

B. Pre–*and post–filtering*;

C. *Balancing*;

D. *The support of a multifilter;*

E. *Symmetry/antisymmetry of multiscaling and multiwavelet functions;*

1.4. Theory of spectral factorization

The spectral factorization has a unique minimum-phase solution $H(e^{j\omega})$ if para-Hermitian polynomial matrix $P(e^{j\omega})$ is positive definite on the unit circle $|z|=1$, absolute integrable with a finite energy and satisfying the Paley-Wiener conditions [152]:

- For scalar spectral factorization

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P(e^{j\omega})d\omega > -\infty \qquad \text{or} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P(e^{j\omega})d\omega < \infty;$$

- For matrix spectral factorization

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det P(e^{j\omega})d\omega > -\infty \qquad \text{or} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det P(e^{j\omega})d\omega < \infty.$$

Therefore, the spectral factorization is the process of determining the minimum phase function belonging to a given power spectrum $P(e^{j\omega})$ which is the product of two factors, H(z), and H^*(z), either of which are to be determined, i.e. $P(e^{j\omega}) = H(z)H^*(z)$.

The fundamental theorem of the spectral factorization is Fejér – Riesz theorem for positive definite functions. The terminology comes from prediction theory, where the nonnegative function $\nu(z)$ plays the role of a spectral density for a multidimensional stationary stochastic process. Fejér [71] first shows the importance of the class of trigonometric polynomials that admit only positive real values; as a theorem it is proved by Riesz [114]. The Fejér – Riesz theorem for a trigonometric univariate polynomial is:

$$v(z)=\sum_{k=-N}^{N} v_k z^k \qquad (1.16)$$

When the function $v(z)$ is real for all $z \in$ T, then the coefficients for all k satisfy the following equality $\bar{v}=v_{-k}$. If $v(z)\geq 0$ for all $z \in$T (unit circle), the factorization of the function $v(z)$ is:

$$v(z)=p(z)p^*(z) \qquad (1.17)$$

where $p(z)=\sum_{k=0}^{N} p_k z^k$ is called a scalar spectral factor and $p^*(z)$ is a Hermitian polynomial.

The scalar spectral factor $p(z)$ is unique to a unitary matrix multiplier $U(z)$ [59], [60], i.e.,

$$p_{new}(z)=p(z)U(z)\,. \qquad (1.18)$$

The Wilson – Burg method [142] for spectral factorization constructs a minimum-phase signal from its spectrum. This is an iterative method with good numerical convergence [58]. Its main application is in study of analytic and bounded functions whose zeros do not lie on the unit circle.

The matrix spectral factorization plays a crucial role in different applications that arize in MIMO systems and control theory [82],[141], [148]. The Bauer's method is well–known for spectral factorization [14], and the implementation of Youla and Kazanjian [102], [103], [109], [149] has been successfully applied [30], [54], [78], [81].

1.5 Theory of spline basis functions

The author has constructed - Linear B–splines; reduced support Hermitian spline for an interval [0,1]; a cubic Hermitian spline ; and a Quintic Hermitian spline.

1.6. Results and conclusions

Based on the identified problems, the author makes a contributionto the theory of wavelets and multiwavelets. For this purpose, a new universal unitary matrix structure was developed, applied in the construction of a new parametric structure model for directly constructing scaling and wavelet functions.

From the overview it follows that scalar and/or vector filter banks can be constructed from Bernstein polynomials, Legendre polynomials, cubic and quintic Hermitian polynomials, as well as linear and quadratic B -splines.

CHAPTER 2
CONSTRUCTION METHODS OF SCALING AND MULTISCALING FUNCTIONS FROM POLYNOMIALS AND SPLINES

2.1 Basis change method

The basis change method for construction of scaling and multiscaling functions consists of constructing a coefficients matrix of the basis functions for a defined interval. This is a product of a scaling function $\phi(t)$ or a multiscaling $\Phi(t)$ function with a nonsingular matrix A, i.e. $A^{-1}A = AA^{-1} = I$. In the multiscaling function case it leads to a new function:

$$\tilde{\Phi}(t) = A\Phi(t) = \sqrt{2}\sum_k AC_k\Phi(2t-k)\,. \tag{2.1}$$

Changing the basis in (2.1) can be represented as:

$$\begin{aligned}\tilde{\Phi}(t) &= \sqrt{2}\sum_k (AC_k \underbrace{A^{-1})A}_{I}\Phi(2t-k) \\ &= \sqrt{2}\sum_k \underline{\mathbf{H}}_k\Phi(2t-k)\end{aligned}\,. \tag{2.2}$$

where *supp* $\Phi(t)$ = *supp* $\tilde{\Phi}(t)$, and the new matrix of coefficients is determined by:

$$\underline{\mathbf{H}}_k = AC_kA^{-1}\,. \tag{2.3}$$

The basis change method is used for the design of multiscaling functions from - Legendre polynomials and *B*–spline, cubic and quintic Hermitian splines.

2.2 Brute force method

The brute force method is a direct method for the design of scaling and multiscaling functions consisting of equalization of the basis function with a scaling function $\phi(t)$ or a multiscaling $\Phi(t)$ function over a defined interval. To apply this method we need:

(a) to divide the basic functions of :

left (L): $\Phi_L(t)$ and right (R): $\Phi_R(t)$.

(b) divide the support of the basis functions of subintervals.

The matrix coefficients from the multiscaling function is found for each subinterval. Example, for the multiscaling function

$$\Phi(t) = \sqrt{2}\left(C_0\Phi(2t) + C_1\Phi(2t-1) + C_2\Phi(2t-2)\right)$$

each matrix coefficient is found from the matrix equations

$$\Phi(t) = \sqrt{2}C_0\Phi(2t)\,, \qquad \text{when t} \in [0,1/2], \tag{2.4}$$

$$\Phi(t) = \sqrt{2}C_0\Phi(2t) + \sqrt{2}C_1\Phi(2t-1)\,, \quad \text{when t} \in [1/2, 3/2] \tag{2.5}$$

$$\Phi(t) = \sqrt{2}C_2\Phi(2t-2)\,, \qquad \text{when t} \in [1/2, 1]. \tag{2.6}$$

2.3 Inner product method

The inner product method uses tensor products and an integral depending on the interval of the basis functions. A disadvantage of the method is its computational complexity.

2.3.1 "*Hat*" scaling function

The support of the multiscaling "*Hat*" function is [0,2]

$$\phi(t) = \sqrt{2}(h_0\phi(2t) + h_1\phi(2t-1) + h_2\phi(2t-2)).$$

The scalar coefficients is determined in the dependent the neighbouring coefficients:

- h_0 dependent from h_1;
- h_1 dependent from h_0 и h_2;
- h_2 dependent from h_1.

This leids to construction and solving of the linear systems:

$$[a, \quad b, \quad c] = \sqrt{2}[h_0, \quad h_1, \quad h_2]\begin{bmatrix} q & r & 0 \\ r & q & r \\ 0 & r & q \end{bmatrix}. \tag{2.7}$$

where three scalar coefficients are:

$$[h_0, \quad h_1, \quad h_2] = \frac{1}{\sqrt{2}}\left[\frac{1}{4}, \quad \frac{5}{12}, \quad \frac{1}{4}\right]\left(\frac{1}{12}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\right)^{-1} = \left[\frac{1}{2\sqrt{2}}, \quad \frac{1}{\sqrt{2}}, \quad \frac{1}{2\sqrt{2}}\right].$$

2.3.2 Multiscaling function of cubic Hermitian spline

The support of the multiscaling function for cubic Hermitian spline is [0,2]

$$\Phi(t) = \sqrt{2}C_0\,\Phi(2t) + \sqrt{2}C_1\,\Phi(2t-1) + \sqrt{2}C_2\,\Phi(2t-2)$$

where C_0, C_1, and C_2 are matrix coefficients. By dividing cubic Hermitian basis functions of '*left*' and 'rigth':

- *left* (L) for the interval $t \in [0,1]$: $\Phi_L(t) = \begin{pmatrix} 3t^2 - 2t^3 \\ t^3 - t^2 \end{pmatrix}$, (2.8)

- *rigth* (R) for the interval $t \in [1,2]$: $\Phi_R(t) = \begin{pmatrix} 3(2-t)^2 - 2(2-t)^3 \\ (2-t)^3 - (2-t)^2 \end{pmatrix}$. (2.9)

is derived the matrix equation:

$$[C_0, \quad C_1, \quad C_2] = \frac{1}{\sqrt{2}}[\mathrm{A} \quad \mathrm{B} \quad \mathrm{C}]\begin{bmatrix} \mathrm{Q} & \mathrm{R} & 0 \\ \mathrm{R}^{\mathrm{T}} & \mathrm{Q} & \mathrm{R} \\ 0 & \mathrm{R}^{\mathrm{T}} & \mathrm{Q} \end{bmatrix}^{-1}. \tag{2.10}$$

whose solutions are the matrix coefficients:

$$C_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}, \; C_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \text{ и } C_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix}. \tag{2.11}$$

2.3.3 Alpert multiscaling function

The support of the Alpert multiscaling function is [0,1]:

$$\Phi(t) = \sqrt{2}C_0\Phi(2t) + \sqrt{2}C_1\Phi(2t-1) \; .$$

where C_0 and C_1 are matrix coefficients, and whose basis functions lie in different intervals. From the inner products:

$$\begin{aligned} \langle\langle \Phi(t), \Phi(2t) \rangle\rangle &= \sqrt{2}C_0 \langle\langle \Phi(2t), \Phi(2t) \rangle\rangle \\ \langle\langle \Phi(t), \Phi(2t-1) \rangle\rangle &= \sqrt{2}C_1 \langle\langle \Phi(2t-1), \Phi(2t-1) \rangle\rangle \end{aligned} \tag{2.12}$$

the two matrix coefficients can be determined

$$C_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \text{ and } C_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}. \tag{2.13}$$

2.4 Results and conclusions

Chapter 2 discusses research opportunities in constructing new methods for constructing banks of polynomials and splines. Thus, three methods for constructing scaling and/or multiscaling functions have been developed. They have been applied to Legendre polynomials, cubic and quintic Hermitian polynomials, and linear and quadratic B-splines.

CHAPTER 3
METHOD OF SPECTRAL FACTORIZATION

3.1 Design of the product filter

The most important part when using the method of spectral factorization for find desirable scaling or multiscaling function that satisfies the *mandatory smoothness condition* of scalar or matrix filter product

$$\det P(z) = \left(\frac{1+z^{-1}}{2}\right)^m Q(z) \tag{3.1}$$

where $Q(z)$ is a linear phase polynomial. The matrix product filter

$$P(z) = P_{-k}z^{-k} + P_{-k+1}z^{-k+1} + \cdots + P_0 + \cdots + P_{k-1}z^{k-1} + P_k z^k$$

satified the *half-band filter condition*:

$$P(z) + P(-z) = 2I \,, \tag{3.2}$$

which means that

$$P_0 = I, \text{ и } P_{2k} = 0, k \neq 0 \,. \tag{3.3}$$

The simplest matrix product filter is the two channel product filter $P(z) \in C^{r\times r}[z, z^{-1}]$ of an first order, i.e. $k = 1$ [89]:

$$P(z) = P_1^T z^{-1} + P_0 + P_1 z \,, \tag{3.4}$$

To achive smoother scaling or multiscaling functions we need higher-order scaling or matrix product filters. This is equivalent to the multiplyer $(1+z)^m$ in the *mandatory smoothness condition* and singular matrix polynomial.

3.2 Quadratic equation method

The method use solving of quadratic equations [89]. Example, for scalar spectral factorization of product filter $p(z)$

$$\begin{aligned} p(z) &= p_{-k}z^{-k} + \cdots + p_{-2}z^{-2} + p_{-1}z^{-1} + p_0 \\ &\qquad\qquad + p_1 z + p_2 z^2 + \cdots + p_k z^k \\ &= h(z)h^*(z) \\ &= (h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots + h_k z^{-k})(h_0 + h_1 z + h_2 z^2 + \cdots + h_k z^k) \end{aligned} \tag{3.5}$$

where $h(z)$ is scalar spectral factor and $h^*(z)$ is scalar Hermitian spectral factor.

3.3 Roots method (Wiener-Hopf factorization)

Roots method [31], [129] use find of polynomial roots:

$$\begin{aligned} p(z) &= h(z)h(z^{-1}) \\ &= p_k \prod_{i=1}^{m}(z - z_i)(z - \frac{1}{z_i})\prod_{j=1}^{k-m}(z - z_j)^2 \end{aligned} \tag{3.6}$$

where $p_k \neq 0$ and Hermitian root z^* is root when z is also root.

3.4 Cepstral method

The cepstral method is based of FFT [28], [33], [83], [94], [110] as can be use to construction of nonsymmetrical orthogonal scaling function. The main idea is logarithm of the scalar filter product:

$$\log p(z) = \sum_{-\infty}^{\infty} l_n z^{-n} = \left(\frac{l_0}{2} + \sum_{1}^{\infty} l_n z^{-n}\right) + \left(\frac{l_0}{2} + \sum_{1}^{\infty} l_n z^{n}\right). \tag{3.7}$$

The factorization consists from the sum of two polynomials as well as iterative finding the coefficients of spectral factor:

$$\begin{aligned} h_0 &= \exp\left(\frac{1}{2} l_0\right), \\ h_n &= l_n h_0 + \frac{n-1}{n} l_{n-1} h_1 + \cdots + \frac{1}{n} l_1 h_{n-1} \end{aligned} . \tag{3.8}$$

3.5 Bauer's method

The Bauer's method is based on the Fejer-Riesz theorem for matrix case [29], [60], [80], [120], [122] which means that spectral factorization of scalar $p(z)$ product filter with coefficients $p_{-k} = p_k$, or (matrix) $P_{-1} = P_1^T$ with product filter $P(z)$ is obtained by Choleski decomposition of the block-band matrix:

$$\begin{aligned} T_{n\times n} &= \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & P_{-k} & P_{-k+1} & \cdots & P_{k-1} & P_k & & \\ & & P_{-k} & P_{-k+1} & \cdots & P_{k-1} & P_k & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} = FF^T \\ &= \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & C_k^{(n)} & \cdots & C_1^{(n-1)} & C_0^{(n-1)} & & \\ & & C_k^{(n)} & \cdots & C_1^{(n-1)} & C_0^{(n-1)} & \\ & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & C_k^{(n)} & \cdots & C_1^{(n-1)} & C_0^{(n-1)} & & \\ & & C_k^{(n)} & \cdots & C_1^{(n-1)} & C_0^{(n-1)} & \\ & & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}^T \end{aligned} \tag{3.9}$$

Consequently, the Bauer's method for spectral factorization is equivalent to the Choleski decomposition of $(n+1)\times(n+1)$ block–band Toeplitz matrix [23], [24]:

$$T_{n\times n}=\begin{bmatrix} P_0 & P_1 & \cdots & P_k & & & \\ P_{-1} & P_0 & P_1 & \cdots & P_k & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \\ P_{-k} & & & \ddots & & & P_k \\ & & & & & & \vdots \\ & & \ddots & & & \ddots & P_1 \\ & & & P_{-k} & \cdots & P_{-1} & P_0 \end{bmatrix}$$

$$F_{n\times n}F_{n\times n}^T=\begin{bmatrix} C_0^{(0)} & & & & & \\ C_1^{(1)} & C_0^{(1)} & & & & \\ \vdots & & \ddots & & & \\ C_k^{(m)} & & & C_0^{(m)} & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & \ddots & \\ & & & C_k^{(n)} & \cdots & C_1^{(n)} & C_0^{(n)} \end{bmatrix}\begin{bmatrix} C_0^{(0)} & & & & & \\ C_1^{(1)} & C_0^{(1)} & & & & \\ \vdots & & \ddots & & & \\ C_k^{(m)} & & & C_0^{(m)} & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & \ddots & \\ & & & C_k^{(n)} & \cdots & C_1^{(n)} & C_0^{(n)} \end{bmatrix}^T. \tag{3.10}$$

The scalar (matrix) spectral factor from the last row is definited:

$$H^{(n)}(z)=C_0^{(n)}+C_1^{(n)}z^{-1}+\cdots+C_k^{(n)}z^{-k}. \tag{3.11}$$

The main disadvantage is its sublinear convergence for singular polynomials (matrix product filters).

3.6 Results and conclusions

Chapter 3 includes an overview of scalar spectral decomposition methods, a development for the matrix case of the smoothness condition for a scalar product filter. The product filter of Alpert multiwavelet filter is developed.

CHAPTER 4
ALGORITHMS FOR BAYER'S METHODS

4.1 General theory of the fast Bauer's method

The advantages of the fast Bauer's method are avoiding the Choleski decomposition of a Toeplitz matrix of an enormous size and the obtaining of a spectral factor with exact values using well-known software tools such as *Matlab* [92], [93] or *Maple* [90]. For the purpose, it is necessary to reduce the degree of the product filter to a first order and solve a NME. Consequently, the *k*–order product filter by rearranging the matrix coefficients P_k

$$P(z) = P_{-k}z^{-k} + P_{-k+1}z^{-k+1} + \cdots + P_0 + \cdots + P_{k-1}z^{k-1} + P_k z^k, \quad \text{and } P_{-1} = P_1^T. \tag{4.1}$$

is a construction a new product filter of order one, i.e. k = 1.

$$\hat{P}(z) = \hat{P}_{-1}z^{-1} + \hat{P}_0 + \hat{P}_1 z. \tag{4.2}$$

where $\hat{P}_0$, and $\hat{P}_{-1} = \hat{P}_1^T$ are the new matrix coefficients.

4.2 Algorithms for the fast Bayer's method

The fast Bauer's method is based on representing the Cholesky factorization of a Toeplitz matrix iteratively row-by-row, which leads to the nonlinear matrix equation (NME) [89], [90]:

$$X^{(n+1)} = P_0 - P_1^T \left[X^{(n)}\right]^{-1} P_1 \tag{4.3}$$

where from the solution $X^{(0)} = P_0$, we find that

$$X^{(t)} = C_0^{(t)}[C_0^{(t)}]^T. \tag{4.4}$$

Since when $n \to \infty$, $X^{(n)} \to X$, then

$$X = P_0 - P_1^T X^{-1} P_1 \tag{4.5}$$

On the basis of the previous theory are constructed *Algorithm 1* (Fig. 4.1) and *2* (Fig.4.2) for finding of scalar(matrix) spectral factors with exact values by Bauer's method for spectral factorization.

4.3 Numerical methods for fast Bauer method

The chapter considers the three methods solving the NME–FPI method, Newton method and GDARE.

4.4 Fast Bauer method (scalar case)

This section considers the fast Bauer method for scalar spectral factorization.

4.5 Fast Bauer method (vector case)

In this section using the fast Bauer method for matrix spectral factorization we obtain Alpert multiscaling and multiwavelet functions.

4.6 Results and conclusions

Chapter 4 is devoted to developing Algorithms 1 and 2 for the fast Bauer method and solving them numerically according to three numerical methods. This is imposed by the main drawback of the classical Bauer method for spectral decomposition – the need to form Cholesky decomposition of a block Toeplitz matrix.

Inputs: $P_0, P_1, \ldots, P_k$ (matrix coefficients of $P(z)$)

Outputs: $C_0, C_1, \ldots, C_k$ (matrix coefficients of $H(z)$)

Begin:

If k >1

Construct block matrices $\hat{P}_0$, $\hat{P}_1$ to reduce the order to one, k= 1;

end

Step 1: Find the matrix $\hat{X}$ by solving ($\hat{X} = P_0 - P_1^T \hat{X}^{-1} P_1$) numerically;

Step 2: Find the matrix $\hat{C}_0$ as the Cholesky factor of $\hat{X}$ ($\hat{X} = C_0 C_0^T$);

Step 3: Find the matrix $\hat{C}_1$ from $\hat{C}_1 = \hat{P}_1 \hat{C}_0^{-T}$;

If k >1

Extract $C_0, C_1, \cdots C_k$ from $\hat{C}_0$ и $\hat{C}_1$;

end

End

Fig. 4.1 *Algorithm 1:* Fast Bauer's method

Inputs: $P_0, P_1, \ldots, P_k$ (matrix coefficients of $P(z)$)

Outputs: $C_0, C_1, \ldots, C_k$ (matrix coefficients of $H(z)$)

Begin:

If k >1

Construct block matrices $\hat{P}_0$, $\hat{P}_1$ so that k = 1;

End

Using a suitable computer algebra system

Step 1: Set up a symmetric matrix $\hat{X}$ with symbol entries x_{ij} ;

Step 2: Set up and solve the nonlinear systems of equations:

$$f(\hat{X}) = \hat{X} - \hat{P}_0 + \hat{P}_1^T \hat{X}^{-1} \hat{P}_1 = 0 ;$$

Step 3: Find the matrix $\hat{C}_0$ as the Cholesky factor of $\hat{X}$ ($\hat{X} = C_0 C_0^T$);

Step 4: Find the matrix $\hat{C}_1$ from $\hat{C}_1 = \hat{P}_1^T \hat{C}_0^{-T}$;

If k >1

Extract $C_0, C_1, \cdots C_k$ from $\hat{C}_0$ и $\hat{C}_1$;

End

End

Fig. 4.2 *Algorithm 2:* Exact Bauer's method

The method is further complicated when finding a desired multiscaling function from a singular filter product, since despite the huge dimensions of the Toeplitz matrix (>10^6) the convergence is even sublinear. The main advantage of BMB is the compilation and solution of NME with - FPI and Newton's methods. A major advantage and application of the developed *Algorithm* 2 is finding desired orthogonal multiscaling functions with exact coefficients. This is verified by constructing orthogonal Alpert multifilter banks.

CHAPTER 5
APPLICATIONS OF METHODS FOR CONSTRUCTIONS OF WAVELET AND MULTIWAVELET FILTER BANKS

5.1 Applications of Bauer's methods for spectral factorization

5.1.1 Haar scaling function

The Haar product filter is singular k = 1 order para – Hermitian polynomial on the unit circle ($z=-1$) with two double zeros,

$$p(z) = p_{-1}z^{-1} + p_0 + p_1 z \tag{5.1}$$

where the coefficients are $p_0 = 1, p_{-1} = \frac{1}{2}$. After applying the exact Bauer's method for spectral factorization the spectral factor is:

$$h_0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} z^{-1}. \tag{5.2}$$

An important charachteristic and essential disadvantage of Bauer's method is the convergence of the diagonal value in block – band Toeplitz matrix:

$$\sqrt{1+\frac{1}{n}} \approx 1 + \frac{1}{2n} \tag{5.3}$$

with an absolute error $\varepsilon_{Haar} = \left| \sqrt{1+\frac{1}{n}} - \left(1+\frac{1}{2n}\right) \right|$ which means that to achieve the desired precision:

(a) $\approx 10^{-9}$ the size of $H_{n\times n}$ needs to be:

$$n = 10^4 \to \varepsilon_{Haar} = \left| \sqrt{1+\frac{1}{10000}} - \left(1+\frac{1}{20000}\right) \right| = 1.3\times 10^{-9}.$$

(б) $\approx 10^{-15}$ the size of $H_{n\times n}$ needs to be:

$$n = 10^7 \to \varepsilon_{Haar} = \left| \sqrt{1+\frac{1}{10^7}} - \left(1+\frac{1}{2\times 10^7}\right) \right| = 1.25\times 10^{-15}.$$

Then, Bauer's method (classical version) is considered for spectral factorization of the Haar product filter with size n= 5-65×10³ of block – band Toeplitz matrix. The numerical errors of the spectral factor and the product filter are shown in fig.5.1 and are obtained by:

$$\begin{aligned} \varepsilon_h &= \| h(z) - h^{(n)}(z) \|_\infty \\ \varepsilon_p &= \| p(z) - h^{(n)}(z) h^{(n)}(z^{-1}) \|_\infty \end{aligned} \tag{5.4}$$

where $h^{(n)} = h_0^{(n)} + h_1^{(n)} z^{-1}$.

5.1.2 Daubechies scaling function

The Daubechies 4 product filter

$$p(z) = p_3(z^{-3} + z^3) + p_2(z^{-2} + z^2) + p_1(z^{-1} + z) + p_0$$

is a singular k = 3 order para – Hermitian polynomial on the unit circle with quadruple zeros $z=-1$):

$$p(z)=h(z)h(z^{-1})=\frac{1}{16}(-z^{-3}+9z^{-1}+16+9z-z^{3}) \\ =-\frac{(1+z)^{4}(z^{2}-4z+1)}{16} . \tag{5.5}$$

According to the exact Bauer's method the well-known spectral factor is:

$$h(z)=\frac{1}{4\sqrt{2}}[(1+\sqrt{3})+(3+\sqrt{3})z^{-1}+(3-\sqrt{3})z^{-2}+(1-\sqrt{3})z^{-3}). \tag{5.6}$$

After applying Bauer's method for a Toeplitz matrix of size n = 58750 the numerical error of the product filter is $\varepsilon_p \approx 1.793\times10^{-10}$, while of the spectral factor is $\varepsilon_h \approx 1.534\times10^{-5}$ even for n = 65000.

5.1.3 Alpert multiscaling function

The Alpert matrix product filter is a singular k = 1 order para – Hermitian polynomial on the unit circle ($z=-1$) with quadruple zeros, i.e. $\det P(z)=\frac{(1+z)^{4}}{16z^{2}}$. After applying of Bauer's method for a size of Toeplitz matrix $n>10^4$ the error of matrix spectral factor is $\varepsilon_H \approx 0.53\times10^{-6}$ for a Toeplitz matrix of size $n > 10^4$.

5.1.4 Fast Bauer method

5.1.4.1 Scalar spectral factorization

A) Haar scaling function

The Haar product filter given in (5.1) has is with coefficients equal to $p_0=1$, $p_1=p_{-1}=1/2$. Then, the solution of the nonlinear scalar (matrix) equation (NME)

$$x=p_0-p^{2}x^{-1} \\ =1-\frac{1}{4}x^{-1} \tag{5.7}$$

is $x=\frac{1}{2}$. By using the Cholesky decomposition of the solution we find that

$$x=h_0h_0=h_0^{2} \\ h_1=p_1^{T}h_0^{-1}=p_1h_0^{-1}$$

which leads to the normalized Haar coefficients:

$$h_0=\sqrt{x}=\frac{1}{\sqrt{2}} \\ h_1=p_1h_0^{-1}=\frac{1}{\sqrt{2}} . \tag{5.8}$$

The fast Bauer method is fast, simplie, and elegant. Moreover, the spectral factor can be found with exact values.

A) Daubechies 4 scaling function

Now, let us investigate the fast Bauer method for scalar spectral decomposition of higher order product filter. Such an example is the Daubechies 4 scaling function.

Obvious, the function is supported in the inteval [0,3]. Thererfore, in order to apply the fast Bauer method it is necessary to reduce the support of product filter to [0,1]. This means that the scalar product filter coefficients will be restructured into two matrix product coefficients, i.e.:

$$p(z) = \tilde{p}_1 z^{-1} + \tilde{p}_0 + \tilde{p}_1 z \tag{5.9}$$

където $\tilde{p}_0 = \begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_0 & p_1 \\ p_2 & p_1 & p_0 \end{bmatrix}$ и $\tilde{p}_1 = \begin{bmatrix} p_3 & p_2 & p_1 \\ 0 & p_3 & p_2 \\ 0 & 0 & p_3 \end{bmatrix}$.

The result is that finding scalar spectral factor with exact values becomes impossible. This is a result of the singularity of the matrix product filter with its quadruple zeros.

5.1.4.2 Fast Bauer method for spectral factorization

(A) Using Algorithm 2

The seven examples with different singularity (Table 5.4) of the scalar and matrix product filters are considered by applying the fast and the exact Bauer methods. The obtained spectral factors are with exact values. In addition, a new supercompact multiwavelet filter is found with smothness $S_{SUP} = 1.28$ which is better than CL $S_{SUP} = 1.06$.

The accuracy of the resulting matrix coefficients is calculated respectively:

- for coefficients of the scaling or the multiscaling function:

$$\varepsilon_H = \| C_0 - C_0^{(n)} \| \tag{5.10}$$

- for coefficients of the scalar or the vector product filter:

$$\varepsilon_P = \| P_0 - C_0^{(n)}[C_0^{(n)}]^T - C_1^{(n)}[C_1^{(n)}]^T \| \tag{5.11}$$

Table 5.4 The characteristics of scalar case (example 2) and matrix cases (examples 1, 3–7)

Example	Singularity	Zeros of the unit circle
1	No	None
2	Yes	Two double
3	Yes	Two double
4	Yes	Quadruple
5	Yes	Quadruple
6	Yes	Quadruple
7	Yes	Decuple

Example 1: (*Nonsinglar para – Hermitian polynomial*) [95].

This is nonsingular prara – Hermitian matrix polynomial of order -2:

$$P(z) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} z^2$$

which the determinant $\det P(z) = \dfrac{(2z-1)(2-z)}{z}$ have zeros of $\dfrac{1}{2}$ and 2. Since the order is greater than one a new 4×4 matrix polynomial withmatrix coefficients $\hat{P}_0$ и $\hat{P}_1$ is obtained. The solution of NME $X = \frac{1}{17}\begin{bmatrix} 8 & 2 & -2 & 0 \\ 2 & 145 & 8 & -34 \\ -2 & 8 & 9 & 0 \\ 0 & -34 & 0 & 153 \end{bmatrix}$ leads to the matrix spectral factor with exact coefficients:

$$H(z) = \frac{1}{\sqrt{34}}\begin{bmatrix} 4 & 0 \\ 1 & 17 \end{bmatrix} + \frac{1}{\sqrt{34}}\begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix} z^{-1} + \frac{1}{\sqrt{34}}\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} z^{-2}.$$

Example 2: (*Haar scaling function*).

The scalar filter product

$$P(z) = z^{-1} + 2 + z = (1+z)(1+z^{-1}) = \frac{(1+z)^2}{z}$$

has double zeros (z = -1). The solution of NME is $x = 1$ which leads to nonnormalized Haar scaling function

$$H(z) = 1 + z^{-1}.$$

Example 3: (*Singular matrix polynomial*) [61].

The singular matrix polynomial

$$P(z) = \begin{bmatrix} 6 & 22 \\ 22 & 84 \end{bmatrix} z^{-1} + \begin{bmatrix} 2 & 7 \\ 11 & 38 \end{bmatrix} + \begin{bmatrix} 6 & 22 \\ 22 & 84 \end{bmatrix} z$$

for which $\det P(z) = -\frac{(z+1)^2(z-1)^2}{z^2}$, has two double zeros ($z = \pm 1$), leads to the solution

$X = \begin{bmatrix} 1 & 5 \\ 5 & 26 \end{bmatrix}$, from which we construct the matrix spectral factor with exact values:

$$H(z) = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix} z.$$

Example 4: (*Integer Multiwavelet*) [32].

This is para – Hermitian matrix polynomial

$$P(z) = \frac{1}{4}\begin{bmatrix} 4 & -\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 4 & \sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix} z$$

for which $\det P(z) = \frac{(1+z^{-1})^4}{8z^2}$, has quadruple zeros (z = -1). The solution of NME

$X = \frac{1}{4}\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ leads to the multiscaling function:

$$H(z) = \frac{1}{2}\left(\begin{bmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{bmatrix} z^{-1}\right).$$

After multipliying of the spectral factor with $\mathbf{C} = diag(\sqrt{2}, 1)$ the well-known integer multiscaling function is obtained [89]:

$$\tilde{H}(z) = \mathbf{C}H(z) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} z^{-1}$$

as its complementary function is the multiwavelet function

$$\tilde{G}(z) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} z^{-1}.$$

Example 5: (*A new supercompact multiwavelets*).

The orthogonal CL multiscaling function [90]

$$H_{CL} = \begin{bmatrix} 0 & \frac{2+\sqrt{7}}{4} \\ 0 & \frac{2-\sqrt{7}}{4} \end{bmatrix} + \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} z^{-1} + \begin{bmatrix} \frac{2-\sqrt{7}}{4} & 0 \\ \frac{2+\sqrt{7}}{4} & 0 \end{bmatrix} z^{-2}.$$

leads to the CL product matrix filter:

$$P_{CL}(z) = \frac{1}{8}\begin{bmatrix} 4 & 1+\sqrt{7} \\ -(1+\sqrt{7}) & -\sqrt{7} \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 4 & -(1+\sqrt{7}) \\ \sqrt{7}+1 & -\sqrt{7} \end{bmatrix} z$$

for which $\det P_{CL}(z) = \dfrac{(4-\sqrt{7})(1+z)^4}{32z^2}$ and which has quadruple zeros on the unit circle. The solution of the NME $X = C_0 C_0^T$ is the symmertrical nonsingular matrix:

$$\begin{aligned} X &= \frac{1}{8}\begin{bmatrix} 4 & \sqrt{7}+1 \\ \sqrt{7}+1 & 4 \end{bmatrix} \\ &= \frac{\sqrt{2}}{8}\begin{bmatrix} 4 & 0 \\ \sqrt{7}+1 & \sqrt{8-2\sqrt{7}} \end{bmatrix} \frac{\sqrt{2}}{8}\begin{bmatrix} 4 & 0 \\ \sqrt{7}+1 & \sqrt{8-2\sqrt{7}} \end{bmatrix}^T. \end{aligned}$$

Since, the square leads to two solutions

$$\begin{aligned} \sqrt{8-2\sqrt{7}} &= \sqrt{(1-\sqrt{7})^2} \\ &= \sqrt{(\sqrt{7}-1)(\sqrt{7}-1)} = \sqrt{(1-\sqrt{7})(1-\sqrt{7})}. \end{aligned}$$

then *the two new orthogonal multiscaling functions* are:

(M1) $(\sqrt{7}-1)$**:** $H(z) = \dfrac{\sqrt{2}}{8}\left(\begin{bmatrix} 4 & 0 \\ \sqrt{7}+1 & \sqrt{7}-1 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ -\sqrt{7}-1 & \sqrt{7}-1 \end{bmatrix} z^{-1}\right).$

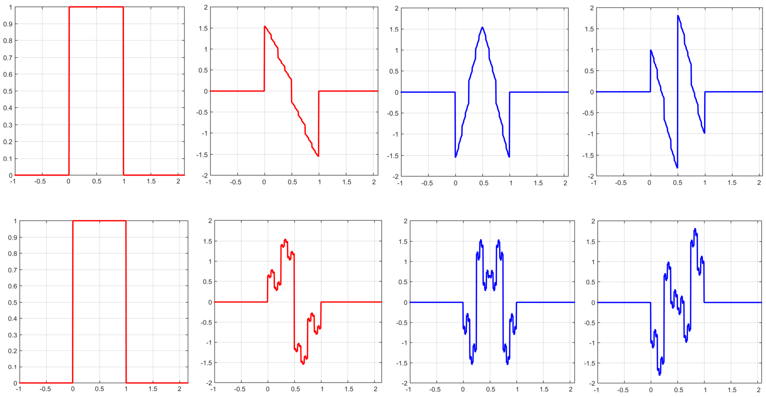
(M2) $(1-\sqrt{7})$**:** $H(z) = \dfrac{\sqrt{2}}{8}\left(\begin{bmatrix} 4 & 0 \\ \sqrt{7}+1 & -(1-\sqrt{7}) \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ -\sqrt{7}-1 & -(\sqrt{7}-1) \end{bmatrix} z^{-1}\right),$

Their multiwavelet functions are found by QR decomposition:

(M1) $G(z) = \dfrac{\sqrt{2}}{8}\left(\begin{bmatrix} 0 & 4 \\ -(1-\sqrt{7}) & -(1+\sqrt{7}) \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 1-\sqrt{7} & -(1+\sqrt{7}) \end{bmatrix} z^{-1}\right).$

(M2) $G(z) = \dfrac{\sqrt{2}}{8}\left(\begin{bmatrix} 0 & 4 \\ -(1-\sqrt{7}) & 1+\sqrt{7} \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 1-\sqrt{7} & 1+\sqrt{7} \end{bmatrix} z^{-1}\right).$

The pairs multiscaling and multiwavelet functions are shown in fig. 5.4.



(a)

(b)

Fig. 5.4 The two new orthogonal supercompact multiscaling functions $\Phi(t)=[\phi_0,\phi_1]^T$ (*red*) and multiwavelet functions $\Psi(t)=[\psi_0,\psi_1]^T$ (blue; (a) (M1) (b) (M2)

Example 6: (*Alpert multiscaling fucntion*).

The singular para – Hermitian product matrix polynomial

$$P(z)=\frac{1}{4}\begin{bmatrix} 2 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} z^{-1}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\frac{1}{4}\begin{bmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} z$$

with $\det P(z)=\dfrac{z^2(1+z)^4}{16}$ has quadruple zeros(z = –1).

From $X=P_0-P_1^T X^{-1} P_1$ and Cholesky decomposition of the solution $X=H_0 H_0^T$ is determined:

$$\text{- } \textit{first coefficient } X=\frac{1}{2}\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{bmatrix};$$

$$\text{- } \textit{second coefficient } H_1=P_1^T H_0^{-T}=\frac{1}{4}\begin{bmatrix} 2 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{bmatrix}^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Example 7: (*Legendre multiscaling function of* order 5) .

The singular para – Hermition matrix polynomial with 5×5 matrix coefficients P_0, P_1, the supercompact multifilter $P(z)=P_0+P_1 z^{-1}$, where $P_0=I$ and

$$P_1=\frac{1}{256}\begin{bmatrix} 128 & 64\sqrt{3} & 0 & -16\sqrt{7} & 0 \\ -64\sqrt{3} & -64 & 16\sqrt{15} & 16\sqrt{21} & -8\sqrt{3} \\ 0 & -16\sqrt{15} & -112 & -8\sqrt{15} & 24\sqrt{5} \\ -64\sqrt{7} & 16\sqrt{21} & 8\sqrt{35} & -40 & -39\sqrt{7} \\ 0 & 8\sqrt{3} & 24\sqrt{5} & 39\sqrt{7} & 53 \end{bmatrix}$$

with $\det P(z)=\dfrac{(z+1)^{10}}{z^5 2^{25}}$ and ten zeros on the unit circle. The solution leads to the multiscaling function:

$$H(z)=\frac{\sqrt{2}}{32}\left(\begin{bmatrix}16&0&0&0&0\\-8\sqrt{3}&8&0&0&0\\0&-4\sqrt{15}&4&0&0\\2\sqrt{7}&2\sqrt{21}&-2\sqrt{35}&2&0\\0&2\sqrt{3}&6\sqrt{5}&3\sqrt{7}&1\end{bmatrix}+\begin{bmatrix}16&0&0&0&0\\8\sqrt{3}&8&0&0&0\\0&4\sqrt{15}&4&0&0\\-2\sqrt{7}&2\sqrt{21}&2\sqrt{35}&2&0\\0&-2\sqrt{3}&6\sqrt{5}&-3\sqrt{7}&1\end{bmatrix}z^{-1}\right)$$

The numerical errors for the product filter in examples 1–7 ε_P and for the spectral factor ε_H, as a result of a fixed-point iteration and Newton's method obtained with *Алгоритъм 1* fast Bauer method are shown in fig.5.5 and fig. 5.6. The results show big differences in the convergence for the two methods.

Б) Using built-in functions

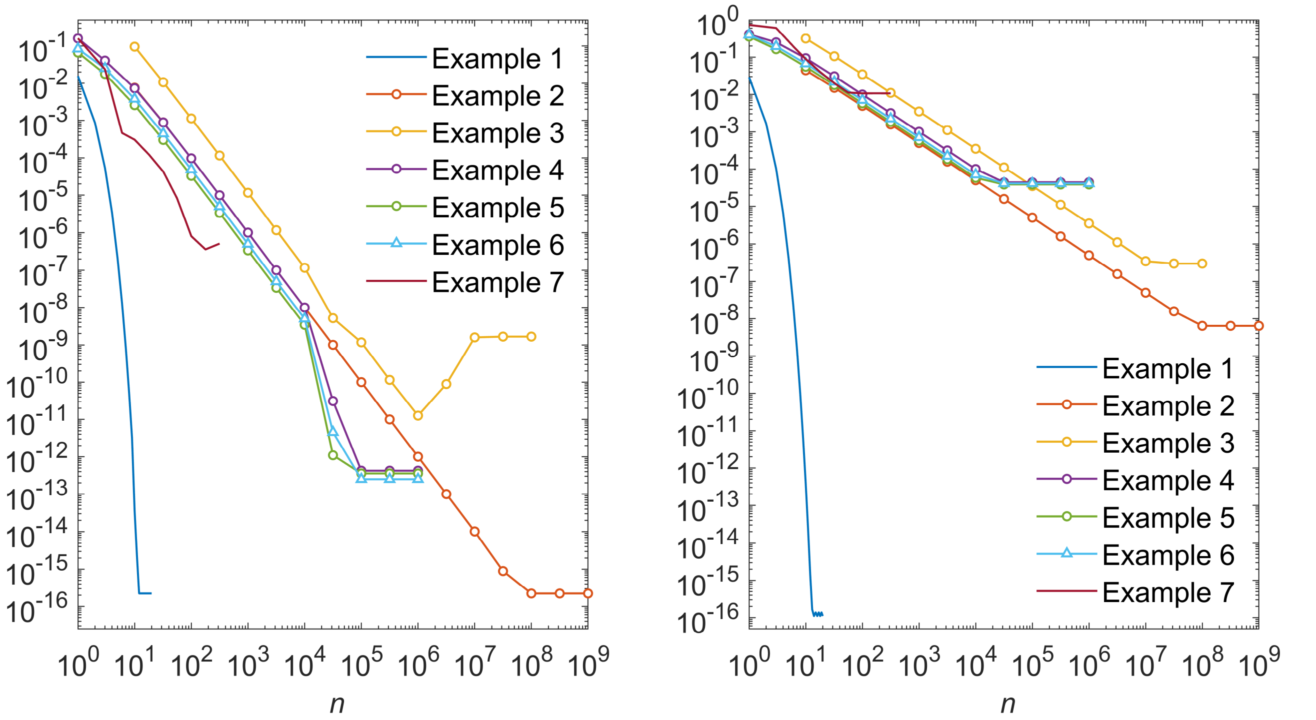
The results from the bult – in functions '*dare*' and '*idare*' for solve of GDARE are:

Maple **-** The results of the numerical errors for solving of GDARE with '*dare*' in *Maple* (Table 5.6) show a high precission of the spectral factorization. Further, obtaining of a high-precission spectral factor (~10^{11}) can be achieved only for nonsingular polynomial (*Example 1*). In cases of para – Hermitian matrix polynomials that possess multiple zeros on the unit circle the achieved precission is very low (~10^4) (*Examples 4–7*). Moreover, in cases of para – Hermitian matrix polynomials that possess double zeros or with different signs (z=±1, Example *3*) on the unit circle, the obtained solution is incorrect. Therefore, using the built-in function is not recommended.

Matlab - Although achieving better accuracy (табл. 5.7) in comparison with *Maple*, for singular para – Hermitian matrix polynomials with multiple zeros, or with double zeros, or with different signs on the unit circle, only the R2018a version provided solutions. Only the precision of the solution of the NME and the spectral factor for example 7 (with decuple zeros) is lower than *Maple*.

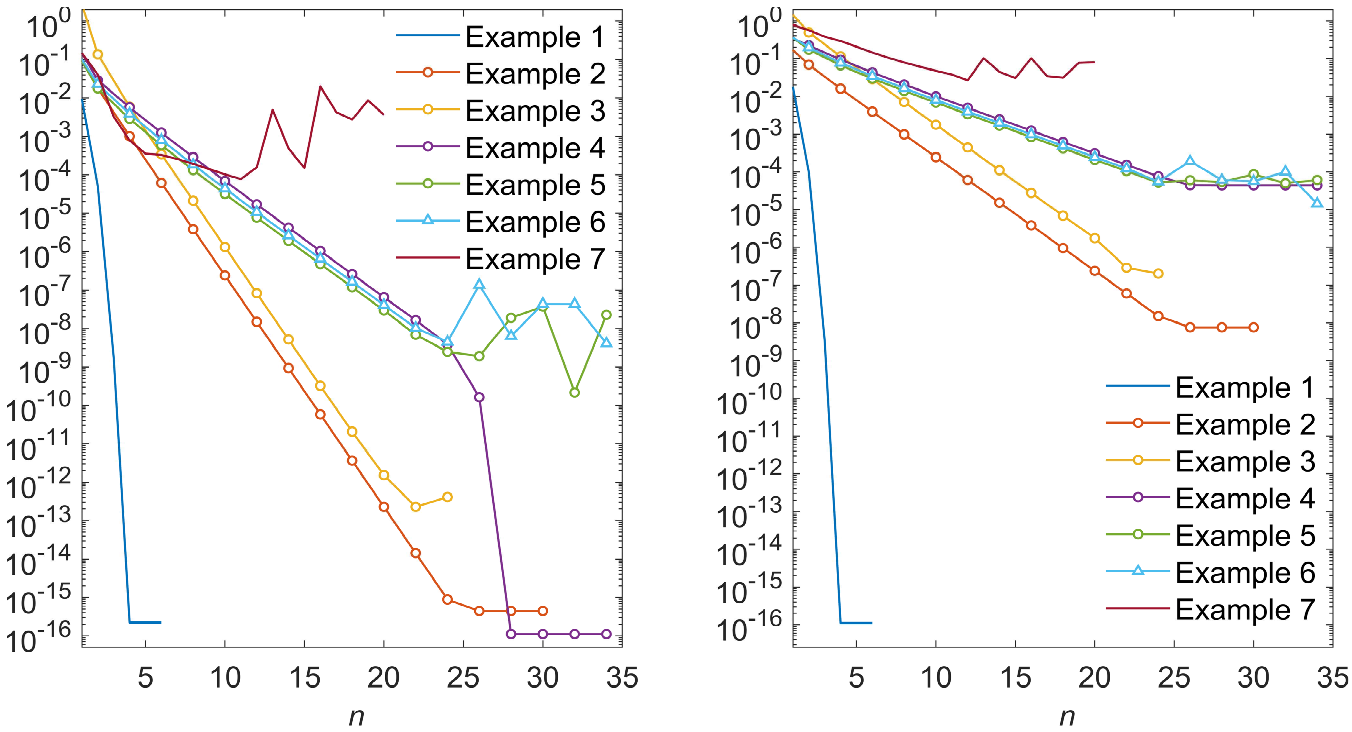
Table 5.6 Errors ε_X, ε_H, and ε_P for examples 1–7, using the '*dare*' in Maple 17

Example	ε_X	ε_H	ε_P
1	2.35e-11	7.01e-11	3.29e-9
2	$p_0 = 2, p_1 = 1$		
	1.22e-8	6.10e-9	0
	$p_0 = 1, p_1 = 1/2$		
	6.10e-9	4.50e-9	5.55e-17
3	Incorrect solution X		
4	5.95e-4	5.95e-4	6.59e-10
5	1.69e-4	2.90e-4	6.59e-10
6	2.20e-4	5.95e-4	2.64e-10
7	2.61e-3	4.41e-2	7.82e-3



(a) (b)

Фиг. 5.5. Log–log plots of the numerical errors obtained by fixed point iteration (FPI) applied to the fast Bauer method (FBM) for scalar and matrix spectral factorization of the product filter. (a) the residuals ε_P (b) the errors ε_H of the spectral factor.



(a) (b)

Фиг. 5.6. Log–log plots of numerical errors obtained by Newton method applied to the fast Bauer method (FBM) for scalar and matrix spectral factorization of the product filter; (a) the residuals ε_P (b) the errors ε_H of the spectral factor.

Таблица 5.7

Nummerical errors ε_X , ε_H , ε_P of examples 1–7 by using build- in functions (‘*dare*’ и ‘*idare*’) for *14* Matlab versions

<table>
<tr><th>Matlab verion</th><th>Example</th><th>ε_X</th><th>ε_H</th><th>ε_P</th></tr>
<tr><td rowspan="9">**R2011a (DARE)**
R2012a (DARE)</td><td>**1**</td><td>1.78e-15</td><td>5.55e-17</td><td>5.90e-16
6.11e-16 (**R2011a**)</td></tr>
<tr><td rowspan="3">**2**</td><td colspan="3">$p_0 = 2$, $p_1 = 1$
No solutions, Report = -1*</td></tr>
<tr><td colspan="3">$p_0 = 1$, $p_1 = 1/2$</td></tr>
<tr><td>4.95e-09</td><td>3.50e-09</td><td>0</td></tr>
<tr><td>**3**</td><td>1.49e-06</td><td>8.31e-07</td><td>5.68e-14</td></tr>
<tr><td>**4**</td><td colspan="3">No solutions. Report = -1*</td></tr>
<tr><td>**5**</td><td colspan="3">No solutions. Report = -1*</td></tr>
<tr><td>**6**</td><td colspan="3">No solutions. Report = -1*</td></tr>
<tr><td>**7**</td><td>0.00105</td><td>0.0137</td><td>1.09e-10</td></tr>
<tr><td rowspan="9">**R2015a (DARE)**
R2016a (DARE)</td><td>**1**</td><td>1.78e-15</td><td>5.55e-17</td><td>4.44e -16</td></tr>
<tr><td rowspan="3">**2**</td><td colspan="3">$p_0 = 2$, $p_1 = 1$
No Solutions. Report = -1*</td></tr>
<tr><td colspan="3">$p_0 = 1$, $p_1 = 1/2$</td></tr>
<tr><td>4.95e-09</td><td>3.50e-09</td><td>0</td></tr>
<tr><td>**3**</td><td>1.49e-06</td><td>8.31e-07</td><td>5.69e-14</td></tr>
<tr><td>**4**</td><td colspan="3" rowspan="3">No Solutions. Report = -1*</td></tr>
<tr><td>**5**</td></tr>
<tr><td>**6**</td></tr>
<tr><td>**7**</td><td>0.00111
(**R2016a**)</td><td>0.0147
(**R2016a**)</td><td>1.53e-11 (**R2016a**)</td></tr>
<tr><td rowspan="9">**R2018a (DARE)**</td><td>**1**</td><td>7.11e-15</td><td>4.16e-17</td><td>1.78e-15</td></tr>
<tr><td rowspan="3">**2**</td><td colspan="3">$p_0 = 2$, $p_1 = 1$
No solutions, Report = -1*</td></tr>
<tr><td colspan="3">$p_0 = 1$, $p_1 = 1/2$</td></tr>
<tr><td>7.00e-09</td><td>4.95e-09</td><td>5.55e-17</td></tr>
<tr><td>**3**</td><td>1.56e-06</td><td>7.98e-07</td><td>9.95e-14</td></tr>
<tr><td>**4**</td><td>5.45e-05</td><td>5.45e-05</td><td>8.88e-16</td></tr>
<tr><td>**5**</td><td>3.58e-05</td><td>6.16e-05</td><td>6.66e-14</td></tr>
<tr><td>**6**</td><td>4.32e-05</td><td>6.12e-05</td><td>4.44e-16</td></tr>
<tr><td>**7**</td><td>0.00108</td><td>0.01418</td><td>1.978e-11</td></tr>
</table>

Таблица 5.7 (continued**)**

Matlab verion	Example	ε_X	ε_H	ε_P
R2019a (IDARE) R2019b (IDARE) R2020a (IDARE) R2020b (IDARE) R2021a (IDARE) R2021b (IDARE) R2022a (IDARE) R2022b (IDARE)	**1**	3.55e-15	1.11e-16	3.55e-15
	2	$p_0 = 2$, $p_1 = 1$ No solutions. Report = -1*		
		$p_0 = 1$, $p_1 = 1/2$		
		5.55e-09	3.92e-09	0
	3	No solutions. Report = 3**		
	4	No solutions. Report = 3**		
	5	4.45e-05	7.64e-05	2.29e-13
	6	4.33e-05	6.12e-05	8.88e-16
	7	0.00101	0.0131	9.50e-11 (R2022a,b) 1.46e-10
R2021a Update 4 (IDARE)	**1**	5.33e-15	8.88e-16	3.55e-15
	2	$p_0 = 2$, $p_1 = 1$ No solutions. Report = 3**		
		$p_0 = 1$, $p_1 = 1/2$		
		No solutions. Report = 3**		
	3	8.54e-07	6.44e-07	1.56e-13
	4	No solutions. Report = 3**		
	5	No solutions. Report = 3**		
	6	4.73e-05	6.69e-05	8.88e-16
	7	9.87e-04	0.01277253	2.97e-10

***Legend*:**

*Report -1 – meant "the associated symplectic pencil has eigenvalues on or very near the unit circle";

**Report 3 – means "The symplectic spectrum has eigenvalues on the unit circle";

5.1.5 Comparative Analsysis of Bauer's methods

The main advantage of Bauer's method is the ability to find a spectral factor, and the main disadvantage is the need for the Cholesky decomposition of an n×n Töpletz matrix of large dimensions (more than $n=65\times10^3$). A major advantage of the fast Bauer method in first-order product filter decomposition is the avoidance of the Toeplitz matrix decomposition, which makes the method fast, simple, and elegant. The other important advantage is the exact values of the spectral factor and the product filter. This allows for simplified hardware schemes to implement filter or multi-filter banks. A major drawback of Bauer's fast method is that, due to the strong influence of unit axis roots in scalar spectral decomposition, it is not guaranteed to find a spectral factor with exact values (Daubeschies 4 scaling function).

5.2 Applications of orthogonal multifilter banks

5.2.1 The lifting scheme of Alpert multiwavelet filter bank with diadic approximation of √3 and applied to 2D сигнали

The influence of 2– and 3–bits quantization of √3 = a/b in the lifing scheme is considered for balanced and non balanced Alpert multiwavelet filtter bank. PSNRs for 3–bits in balanced and nonbalanced multifilter lead to high-quality images, while 2–bits quantization lead to big errors. Hence, for nonbalanced filter bank and 2–bits quantization for all decomposition levels $J > 1$ the obtained images have a mesh structure as well as for balanced filter bank for $J \geq 4$ artifacts appear. Therefore, 3–bits quantization is necessary to achieve high-quality image for two types balancing.

5.2.2 Comparative analysis of three orthogonal multifilters for denoising of gray-levels image

Signal denoising is one of the most common applications. It is the shrinking, suppressing, or removal, if possible of the noise.

In this research is consider an input signal *s* with AWGN, $e \sim N(0, \sigma^2)$ and obtain the denoisng signal $\hat{s}$. The technique for denoising the signal s (gray level images) is the multiwavelet transform that possesses the important property that the input signal energy is concentrated in several wavelet coefficients, the noise energy in subbands with Gaussian distribution. The '*hard*' and '*soft*' threshold values are selected. The highest *PSNR* for '*hard*' threshold is obtain for denoising with Alpert multifilter, while for '*soft*' threshold– GHM.

5.2.3 Comparative analysis of orthogonal scalar and vector filters for image compression of astronomical image from scanned photograph plates

The Haar – like, Daubechies - like multiwavelet filter banks [86], [87] with its scalar versions and GHM multifilter are compared for compression of astronomical images from scanned photograph plates. The results shows better quality of compression for such a type images with Haar – and Daubechies like multiwavelets. Extremely effective for lossless compression for images with highly non-smooth areas of uniform intensity is the Haar – like multiwavelet filter bank to 5 decomposition level. A disadvantage of scalar Daubechies filter bank is the existence of structure dependence in the astronomy image from scanned photograph plates.

5.3 Results and conclusions

Chapter 5 shows fast Bauer method compared with the classical method, and the finding of scalar or matrix spectral factor without Cholesky decomposition of $n \times n$ Toeplitz matrix and with exact coefficients.

To avoid unwanted defects in image processing of gray-level images it is quite enough the 3-bit quantization of the coefficient √3 with balanced or nonbalanced Alpert multiwavelet filter bank.

The balanced Alpert multiwavelet filter bank, despite its shorter length and smaller coding gain, achieves a higher PSNR compared to GHM and CL multifilters for image denoising of some test gray levels images. Moreover, for gray image '*Lizard*' the balanced and nonblanced Alpert filter achive better PSNR.

Compression of astronomy images from scanned photograph plates is achieved by using of multifilter version of scalar filters - Haar – like or Daubechies - like multiwavelet filter banks.

CONCLUSION

The dissertation investigates problems related to new methods for developing scaling or multiscaling functions satisfying orthogonal or non-orthogonal conditions.

Four explicit and simple methods have been developed and investigated to prove the hypothesis that multiwavelength filter banks can be constructed from different basis functions (polynomials, splines) and that Bauer's method of spectral decomposition can be used to derive desired multiwavelength filter banks.

The contributions are in the field of wavelet theory. It is possible to consider and apply other polynomials and spline functions not considered in the dissertation.

A big attention needs to be paid to the development of Bauer's method for the matrix spectral decomposition of low degree singular matrix polynomials that avoids the Cholesky decomposition of large Toeplitz matrices. From a mathematical point of view - it solves multidimensional singular matrix polynomials, and from an engineering point of view - it solves the problem of finding orthogonal multiscaling functions with desired smoothness.

Two algorithms of Bauer's spectral decomposition method for singular matrix polynomials are developed for the first time. Its numerical errors using the fixed point and the Newton numerical methods were studied, as well as solutions of NME for 14 versions of Matlab ('*dare*' and '*idare*') and Maple 17 ('*dare*').

Moreover, the obtained matrix spectral factors are with exact values. One example is the Alpert matrix product filter which is constructed for the first time.

Scalar and multiwavelet filter banks of Haar and Daubechies, Haar – like and Daubechies - like, CL, Alpert, and GHM for denoising and compression of gray-level images are considered.

The process of design is based on basic matrix algebra so that it can be easily and conveniently used by both students and researchers.

FUTURE DEVELOPMENT

Modern signal processing is undergoing rapid development in the analog and discrete domains. In particular, the wavelet theory is part of basic research in many fields. Therefore, a new direction is the development and generalization of the theory of spectral matrix decomposition for the multivariable case. This can lead to improved denoising, compression, or mixed signal processing in the big-data area.

Another new direction is the implementation of the lifting scheme, and software or hardware implementations of the orthogonal multiwavelet filter banks.

In this scientific study, the scientific results are applied in the processing of gray-level test images. The investigation and proof of the advantages of the obtained new multi-filter banks in color photos or video processing is another task for future research.

AUTHOR CONTRIBUTIONS

Scientific results

1. Three elegant, simple and fast methods for deriving the matrix recursion coefficients of scalar and vector functions are constructed: basis change, brute force and inner product .
2. For first time a two-channels Alpert matrix product filter is constructed.
3. Two new orthogonal supercompact multiscaling functions are obtained by using fast Bauer method for matrix spectral factorization and their complementary orthogonal supercompact multiwavelet functions are obtined.

Scientific - applied results

1. By using the three methods and Bauer's method of spectral factorization scaling or multiscaling functions from B–spline, qudtaric B–spline, cubic and quintic Hermitian spline and Legendre polynomials are designed.
2. Two new algorithms for fast and exact Bauer methods are constructed.
   (a) *Algorithm 1* – fast Bauer's method;
   (б) *Algorithm 2* – exact Bauer's method;
   The author has constructed two numerical methods (based on the fixed-point iterations technique and based on the Newton method) for solving of singular NMEs. Comparitive analysis of build–in functions '*dare*' и '*idare*' in Maple and Matlab for solving of singular NMEs by Generalized Discrete Time Algebraic Riccati Equation (GDARE) of Seven examples are detail considered in detail.
3. A novel lifting scheme for the Alpert multiwavelet filter bank with two types quantization of coefficient $\sqrt{3}$ is constructed. It is applied in filter banks to gray-level images without additional processing.
4. The мultiplierless modules for analysis and syntesis of a perfectly restorative bio-orthogonal 5/3 filter bank on Xilinx's Virtex and Spartan series are implemented.

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