

Some Results on Weights of Vectors Having m -Repeated Bursts

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Abstract: *In coding theory the importance of burst error correcting and detecting codes is well known. In general communication the messages are long and the strings of bursts may be short, repeating in a vector itself. The idea of repeated bursts, introduced by Beraradi, Dass and Verma has opened this area of study. They defined 2-repeated bursts and obtained results for detection and correction of such type of errors. Later on, Dass and Verma generalized this idea and defined m -repeated bursts. They also obtained results regarding the correction and detection of m -repeated bursts.*

In this paper we obtain results on weights of all vectors having m -repeated bursts of the same size. Another section is devoted to the study of vectors having m -repeated bursts with weight constraints. This study can help developing more efficient codes with these vectors as error patterns.

Keywords: *m -repeated bursts, efficient codes, weights of vectors.*

1. Introduction

It is noticed that in different kind of communication channels there are different types of errors. One of the types which occurs more frequently in many channels is that of burst errors. This led to the study of burst error correcting codes, introduced by Fire [5] and Regier [8], and thoroughly treated by Peterson and Weldon [7]. Stone [11], Bridewell and Wolf [1] considered multiple bursts. Chien and Tang [2] also considered a different type of a burst, known as CT burst. Moreover the study of burst error correcting codes is now becoming more

important from an application point of view because of its easy implementation and efficient functioning.

Yet another kind of an error pattern, called 2-repeated burst has been introduced by D a s s, V e r m a and B e r a r d i [3]. This is an extension of the idea of open-loop burst given by Fire. Later on D a s s and V e r m a [4] defined m -repeated bursts and obtained results regarding the number of parity-check digits required for codes, correcting such errors. While some results have been obtained on bounds for m -repeated burst error correcting codes with specific distance and parity-check digits, but the importance of weight is still untouched.

The study of bursts in terms of weight was initiated by S h a r m a and D a s s [9]. K r i s h n a m u r t h y [6] extended their work by obtaining some combinatorial results regarding the weight of burst error correcting codes. Recently, S h a r m a and R o h t a g i [10] obtained some results on weights of 2-repeated bursts.

In this correspondence we obtain results regarding the weight of all vectors having m -repeated bursts of length b each. This correspondence is organized as follows. Basic definitions, related to our study are stated with some examples in Section 2. In Section 3 some results on m -repeated bursts are derived. In Section 4 combinatorial results on weights of m -repeated bursts are obtained. Thus we generalize the results obtained in our earlier paper [10].

Further on we shall consider the space of n -tuples whose nonzero components are taken from the field of q code characters with elements $0, 1, 2, \dots, q-1$. The weight of a vector is considered in Hamming sense as the number of non-zero entries.

2. Preliminaries

We give the definition of a burst, defined by Fire, as taken in [8].

Definition 2.1. A burst of length b is a vector, all of whose nonzero components are confined to some b consecutive components, the first and the last of which are nonzero.

A vector may have not just one cluster of errors, but more than one. Lumping them into one burst amounts to neglecting the nature of communication and unnecessarily considering a longer burst which may have a part, which is not of cluster in-between. For example, in a very busy communication channel, sometimes bursts repeat themselves. D a s s, V e r m a and B e r a r d i [3] introduced the idea of repeated bursts. In particular, they defined the “*2-repeated burst*”.

A *2-repeated burst* of length b may be defined as follows:

Definition 2.2. A 2-repeated burst of length b is a vector of length n whose only nonzero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being nonzero.

Example: (0120400100300) is a 2-repeated burst of length 4 over GF(5).

An *m-repeated burst* of length b may be defined as follows:

Definition 2.3. An m -repeated burst of length m is a vector of length n whose only nonzero components are confined to m distinct sets of b consecutive components, the first and the last component of each set being nonzero.

Example: (001020024100314030100) is an m -repeated burst of length 3 over GF(5).

Since the weight structure is of considerable interest, in the next section we present some results on weights of m -repeated bursts.

3. Results on weights of m -repeated bursts

Let W_{mb} denotes the total weight of all vectors having m -repeated bursts of length b in the space of all n -tuples. Before obtaining W_{mb} in terms of n and b we derive two results in the lemmas below, counting the m -repeated bursts.

Lemma 3.1. The total number of m -repeated bursts, each of length $b > 1$, in the space of all n -tuples over GF(q) is

$$(1) \quad [q^{(b-2)}(q-1)^2]^m \frac{(n-mb+1)(n-mb+2)}{2}, m \geq 2.$$

P r o o f: Consider an n -tuple in which an m -repeated burst of length b is defined as follows.

An m -repeated burst of length b is a vector whose only nonzero components are confined in m distinct sets of b consecutive components, the first and the last components of each set being nonzero.

To make m -repeated bursts of length b each, the first burst can start from i -th position, where i varies from 1 to $n-mb+1$. The second burst can then start from a position after the first one ends. This process continues till the m -th burst ends.

Let us first consider the vector having m -repeated bursts, in which the first burst starts from the first position, their number being $(q-1)^2 q^{b-2}$, then the second burst, their number also being $(q-1)^2 q^{b-2}$, continuing upto m -th burst their number is also $(q-1)^2 q^{b-2}$, then there will be $n-mb+1$ starting positions. Thus the total number of m -repeated bursts, in which the first burst starts from the first position, is given by

$$[(q-1)^2 q^{b-2}]^m (n-mb+1).$$

Next, considering the vector with m -repeated bursts, in which the first burst starts from the second position, the starting positions of all bursts being reduced by 1, their number will be

$$[(q-1)^2 q^{b-2}]^m (n-mb).$$

A little consideration will show that the process of constructing m -repeated bursts will end when the m -th burst has just one starting position, the number then being $[(q-1)^2 q^{b-2}]^m \cdot 1$.

Summing all, the total number of n -vectors having m -repeated bursts of length b each will be

$$[q^{(b-2)}(q-1)^2]^m \sum_1^{n-mb+1} i = [q^{(b-2)}(q-1)^2]^m \frac{(n-mb+1)(n-mb+2)}{2}.$$

This proves the result.

Next we impose weight restriction on m -repeated bursts and count their numbers. The result is given in the lemma below.

Lemma 3.2. The total number of vectors having m -repeated bursts of length $b > 1$ with weight w ($2m \leq w \leq mb$) in the space of all n -tuples is

$$(2) \quad \binom{mb-2m}{w-2m} (q-1)^w \frac{(n-mb+1)(n-mb+2)}{2}, \quad m \geq 2.$$

Proof: Let us consider a vector having m -repeated bursts of length b each. Its only nonzero components are confined to m distinct sets of b consecutive components, the first and the last component of each set being nonzero. Each of them, the first and the last components may be any of the $q-1$ nonzero field elements. Since we are considering m -repeated bursts of length b , in a vector of length n , having weight w , this will have non-zero positions as follows:

- i. The first and the last position of all m bursts.
- ii. Some $w_1 - 2$ amongst the $b - 2$ in-between positions of the first burst, $w_2 - 2$ amongst the $b - 2$ in-between positions of the second burst, $w_m - 2$ amongst the $b - 2$ in-between positions of m -th burst, where w_1, w_2, \dots, w_m are weights of the first, second, ..., m -th bursts respectively and $w_1 + w_2 + \dots + w_m = w$.

- iii. The other positions have the value 0.

Thus, in combinatorial ways, each m -repeated burst will give its number by

$$(3) \quad (q-1)^{2m} (q-1)^{w-2m} \binom{b-2}{w_1-2} \binom{b-2}{w_2-2} \dots \binom{b-2}{w_m-2}.$$

To find a closed expression for

$$\binom{b-2}{w_1-2} \binom{b-2}{w_2-2} \dots \binom{b-2}{w_m-2},$$

we consider the following identity:

$$(4) \quad (1+x)^{mb-2m} = (1+x)^{b-2} (1+x)^{b-2} \dots \text{of up to } m \text{ terms.}$$

Equating the coefficients of x^{mb-2m} on both sides, we get

$$\binom{mb-2m}{w-2m} = \binom{b-2}{w_1-2} \binom{b-2}{w_2-2} \dots \binom{b-2}{w_m-2}.$$

Using this identity, the total number of m -repeated bursts of length b and weight w , with a sum of their starting position $\frac{(n-mb+1)(n-mb+2)}{2}$, is

$$\binom{mb-2m}{w-2m} (q-1)^w \frac{(n-mb+1)(n-mb+2)}{2}.$$

This proves the lemma ■

Now we return to finding an expression for W_{mb} , the total weight of all vectors having m -repeated bursts of length b in the space of all n -tuples.

Theorem 3.1. For $n \geq b$ and $m \geq 2$

$$(5) \quad W_m = \frac{n(n-1)(n-2)\dots(n-m+1)}{\Gamma(m)} (q-1)^2$$

and

$$(6) \quad W_{mb} = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} q^{(mb-2m-1)} [mb(q-1)+2m].$$

P r o o f: The value of W_m follows simply by considering all vectors having m nonzero entries out of n . Their number

$$\binom{n}{m} (q-1)^2 = \frac{n(n-1)(n-2)\dots(n-m+1)}{\Gamma(m)} (q-1)^2$$

clearly gives the value of W_m .

Next, for $b > 1$, using Lemma 3.2, the total weight of all vectors having m -repeated bursts of length b each, is given by

$$\begin{aligned} & \sum_{w=2m}^{mb} w \binom{mb-2m}{w-2m} \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^w = \\ & = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} \sum_{i=0}^{mb-2m} (i+2m) \binom{mb-2m}{i} (q-1)^i = \\ & = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} \frac{1}{(q-1)^{2m-1}} \frac{d}{dq} [(q-1)^{2m} \{1+(q-1)\}^{mb-2m}] = \\ & = \frac{(n-mb+1)(n-mb+2)}{2} (q-1) \frac{d}{dq} [(q-1)^{2m} q^{mb-2m}] = \\ & = \frac{(n-mb+1)(n-mb+2)}{2} (q-1) [(mb-2m)(q-1)^{2m} q^{mb-2m-1} + q^{mb-2m} \cdot 2m(q-1)^{2m-1}] = \\ & = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} q^{mb-2m-1} [(mb-2m)(q-1) + 2mq] = \\ & = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} q^{mb-2m-1} [mb(q-1) + 2m]. \end{aligned}$$

This completes the proof of the theorem ■

Further, in coding theory, an important criterion is to look for the minimum weight in a group of vectors. Our next theorem is a result in this direction.

Theorem 3.2. The minimum weight of a vector having m -repeated burst of length $b > 1$ in the space of all n -tuples is at most

$$(7) \quad mb - \frac{m(b-2)}{q}, \quad m \geq 2.$$

Proof: From Lemma 3.1 it is clear that the number of m -repeated bursts of length b in the space of all n -tuples with symbols taken from the field of q elements is

$$[q^{(b-2)}(q-1)^2]^m \frac{(n-mb+1)(n-mb+2)}{2}$$

also from Theorem 3.1, their total weight is

$$\frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} q^{mb-2m-1} [mb(q-1) + 2m].$$

Since the minimum weight element can be at most equal to the average weight, an upper bound on minimum weight of an m -repeated burst of length b is given by

$$\begin{aligned} & \frac{2(n-mb+1)(n-mb+2)(q-1)^{2m} q^{mb-2m-1} [mb(q-1) + 2m]}{[q^{(b-2)}(q-1)^2]^m (n-mb+1)(n-mb+2).2} = \\ & = mb - \frac{m(b-2)}{q}. \end{aligned}$$

This proves the result ■

4. Combinatorial results on weights of vectors having m -repeated bursts with a weight constraint

Let $W_{mb,w}$ denotes the total weight of those vectors having m -repeated bursts of length b each, which are of weight w or less in the space of all n -tuples over $GF(q)$. Before obtaining the main results, we state a simple result in the lemma below.

Lemma 4.1. Let $[1+x]^{(n,r)}$ denotes the incomplete binomial expansion

$$1 + \binom{n}{1}x + \dots + \binom{n}{r}x^r \text{ of } (1+x)^n$$

up to the term containing x^r , $r \leq n$, in the ascending powers of x .

Then $\frac{d}{dx}[1+x]^{(n,r)} = n[1+x]^{(n-1,r-1)}$, where $\frac{d}{dx}$ stands for the derivative with respect to x .

Theorem 4.1. In the space of all n -tuples over $\text{GF}(q)$, for $n \geq mb \geq w > 1$ and $m \geq 2$,

$$(8) \quad W_{mb,w} = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} \left[2[1+(q-1)]^{(mb-2m, w-2m)} + (q-1)(mb-2m)[1+(q-1)]^{(mb-2m-1, w-2m-1)} \right].$$

P r o o f: We know from Lemma 3.2 that the total number of vectors having m -repeated bursts of length $b > 1$ each, with weight w in the space of all n -tuples is

$$\binom{mb-2m}{w-2m} (q-1)^w \frac{(n-mb+1)(n-mb+2)}{2},$$

therefore, $W_{mb,w}$ is the total weight of m -repeated bursts of length b each with weight w or less, where $2m \leq w \leq mb$, is given by

$$\begin{aligned} W_{mb,w} &= \sum_{i=2m}^w i \binom{mb-2m}{w-2m} (q-1)^i \frac{(n-mb+1)(n-mb+2)}{2} = \\ &= \frac{(n-mb+1)(n-mb+2)}{2} (q-1) \left[2m(q-1)^{2m-1} + \right. \\ &\quad \left. + (2m+1) \binom{mb-2m}{1} (q-1)^{2m} + \dots + w \binom{mb-2m}{w-2m} (q-1)^{w-1} \right] = \\ &= \frac{(n-mb+1)(n-mb+2)}{2} (q-1) \frac{d}{dq} \left[(q-1)^{2m} + \binom{mb-2m}{1} (q-1)^{2m+1} + \right. \\ &\quad \left. + \dots + \binom{mb-2m}{w-2m} (q-1)^w \right] = \frac{(n-mb+1)(n-mb+2)}{2} (q-1) \frac{d}{dq} \left[(q-1)^{2m} \left\{ 1 + \right. \right. \\ &\quad \left. \left. + \binom{mb-2m}{1} (q-1) + \dots + \binom{mb-2m}{w-2m} (q-1)^{w-2m} \right\} \right] = \\ &= \frac{(n-mb+1)(n-mb+2)}{2} (q-1) \frac{d}{dq} \left[(q-1)^{2m} [1+(q-1)]^{(mb-2m, w-2m)} \right] \end{aligned}$$

and using Lemma 4.1, we get

$$\begin{aligned} W_{mb,w} &= \frac{(n-mb+1)(n-mb+2)}{2} (q-1) \left[(q-1)^{2m} (mb-2m) [1 + \right. \\ &\quad \left. + (q-1)]^{(mb-2m-1, w-2m-1)} + 2(q-1)^{2m-1} [1+(q-1)]^{(mb-2m, w-2m)} \right] = \\ &= \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} \left[2[1+(q-1)]^{(mb-2m, w-2m)} + \right. \\ &\quad \left. + (q-1)(mb-2m)[1+(q-1)]^{(mb-2m-1, w-2m-1)} \right]. \end{aligned}$$

This proves the theorem ■

Next we give a recurrence relation for weights in this very general case.

Theorem 4.2. A recurrence relation between $W_{mb,w}$ and $W_{mb-1,w-1}$ is given by

$$(9) \quad \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} \frac{d}{dq} \left[\frac{2.W_{mb,w}}{(n-mb+1)(n-mb+2)(q-1)^{2m}} \right] = \\ = (mb-2m)W_{mb-1,w-1} + \\ + \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} (mb-2m)[1+(q-1)]^{(mb-2m-1,w-2m-1)}$$

where $m \geq 2$.

Proof: From Theorem 4.1 we have

$$W_{mb,w} = \frac{(n-mb+1)(n-mb+2)}{2} (q-1)^{2m} \left[2[1+(q-1)]^{(mb-2m,w-2m)} + \right. \\ \left. + (q-1)(mb-2m)[1+(q-1)]^{(mb-2m-1,w-2m-1)} \right].$$

Therefore

$$(10) \quad W_{mb-1,w-1} = \frac{(n-mb+2)(n-mb+3)}{2} (q-1)^{2m} \left[2[1+(q-1)]^{(mb-2m-1,w-2m-1)} + \right. \\ \left. + (q-1)(mb-2m-1)[1+(q-1)]^{(mb-2m-2,w-2m-2)} \right].$$

and

$$(11) \quad \frac{2.W_{mb,w}}{(n-mb+1)(n-mb+2)(q-1)^{2m}} = \left[2[1+(q-1)]^{(mb-2m,w-2m)} + \right. \\ \left. + (q-1)(mb-2m)[1+(q-1)]^{(mb-2m-1,w-2m-1)} \right].$$

Differentiating with respect to q and then using Lemma 4.1, we get

$$\frac{d}{dq} \frac{2.W_{mb,w}}{(n-mb+1)(n-mb+2)(q-1)^{2m}} = \left[2(mb-2m)[1+(q-1)]^{(mb-2m-1,w-2m-1)} + \right. \\ \left. + (q-1)(mb-2m)(mb-2m-1)[1+(q-1)]^{(mb-2m-2,w-2m-2)} \right] + \\ + (mb-2m)[1+(q-1)]^{(mb-2m-1,w-2m-1)} = (mb-2m) \left[2[1+(q-1)]^{(mb-2m-1,w-2m-1)} + \right. \\ \left. + (mb-2m-1)(q-1)[1+(q-1)]^{(mb-2m-2,w-2m-2)} + [1+(q-1)]^{(mb-2m-1,w-2m-1)} \right].$$

The result now follows by substituting the value of $W_{mb-1,w-1}$ ■

Finally we have the result of the upper bound on the minimum weight vector in the class of vectors considered in this section.

Theorem 4.3. The minimum weight of an m -repeated burst of length b with weight w or less in the space of all n -tuples over $\text{GF}(q)$, is at most

$$(12) \quad 2 + \frac{(q-1)(mb-2m)[1+(q-1)]^{(mb-2m-1,w-2m-1)}}{[1+(q-1)]^{(mb-2m,w-2m)}}, \quad m \geq 2.$$

Proof: Using Lemma 3.2, the total number of m -repeated bursts of length b with weight w in the space of all n -tuples over $\text{GF}(q)$ is given by

$$\frac{(n - mb + 1)(n - mb + 2)}{2} (q - 1)^{2m} [1 + (q - 1)]^{(mb - 2m, w - 2m)}.$$

From Theorem 4.1 the total weight is

$$W_{mb,w} = \frac{(n - mb + 1)(n - mb + 2)}{2} (q - 1)^{2m} \left[2[1 + (q - 1)]^{(mb - 2m, w - 2m)} + (q - 1)(mb - 2m)[1 + (q - 1)]^{(mb - 2m - 1, w - 2m - 1)} \right].$$

Since the minimum weight element is at most equal to the average weight, the minimum weight of an m -repeated burst of length b with weight w or less is at most

$$\begin{aligned} W_{mb,w} &= (n - mb + 1)(n - mb + 2)(q - 1)^{2m} [2[1 + (q - 1)]^{(mb - 2m, w - 2m)} + \\ &+ \frac{2(q - 1)(mb - 2m)[1 + (q - 1)]^{(mb - 2m - 1, w - 2m - 1)}}{2(n - mb + 1)(n - mb + 2)(q - 1)[1 + (q - 1)]^{(mb - 2m, w - 2m)}}] = \\ &= 2 + \frac{(q - 1)(mb - 2m)[1 + (q - 1)]^{(mb - 2m - 1, w - 2m - 1)}}{[1 + (q - 1)]^{(mb - 2m, w - 2m)}}. \end{aligned}$$

This proves the result ■

Conclusion

In the study we are examining multiple burst error correction of length b each in Quasi-cyclic codes with shifts in b positions.

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