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Experimental Robust Stability Analysis of Uncertain Dynamic Models[1]

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Abstract: *The robust stability analysis of four uncertain linear models is considered. It is shown, that the application of a newly derived approach, taking into account the inequalities between the entries of the uncertain parameter vector, leads to a considerable decrease in the computational complexity, in comparison with some known results. The efficiency of the respective conditions, aimed at solving the same task is discussed.*

Keywords: *HMP, Lyapunov stability, uncertain linear dynamic systems, robust stability.*

1. Introduction

Robustness of linear systems subjected to structured real parametric uncertainty belonging to a compact vector set (e.g., the unit simplex) has been recognised as a key issue in the analysis of control systems, but robust stability cannot be assessed using convex optimisation. When applying the powerful Lyapunov's second method for parameter dependent Lyapunov functions, it comes out that the efficiency of the respective stability condition depends extremely on the condition for positive definiteness of a special Homogeneous Matrix Polynomial (HMP) of a possibly high degree. This task usually reduces to the formulation of a system of Linear Matrix Inequalities (LMIs). It is important to get a system of highly non-conservative LMIs, which will inevitably result in an efficient analysis procedure, requiring minimal computational resources. In some recent works [10-12], it has

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been shown, that the most widely applied approach to solve the problem is a very conservative one, although it provides an asymptotically exact condition for stability of uncertain linear systems. By means of taking into account the presence of inequalities between the entries of the uncertain vector, the authors succeeded to prove, that conservativeness in the necessary and sufficient condition can be considerably reduced, which finally results in a less complicated computational procedure.

This research work is organized as follows. Problem formulation, including brief analysis of previous results, is done in Section 2. Section 3 contains in a condensed form the theoretical background of the new approach and recalls two well known results, which are used to perform the comparative analysis. The experimental robust stability study of an overhead crane system, electromechanical system, an induction motor and a power system is done in Section 4, where the properties of the separate conditions are discussed. The efficiency of the respective stability conditions is compared and analyzed.

2. Preliminaries, previous results, open problems

The notation $A > 0, (A \geq 0)$ indicates that A is a positive (semi-positive) definite matrix, $A = [a_{ij}] \in \mathbf{R}_n$ and $a = (a_i) \in \mathbf{R}^N$ denote real $n \times n$ matrix and $N \times 1$ vector with entries a_{ij} and a_i, respectively. The sum of N nonnegative scalars α_i is $|\alpha|$. Define also the vector sets $\mathbf{x}_n \equiv \{x \in \mathbf{R}^n : x^{\mathrm{T}} x = 1\}$ and $\boldsymbol{\omega}_N \equiv \{\alpha = (\alpha_i) \in \mathbf{R}^N : |\alpha| = 1\}$. $\mathbf{N}_x$ denotes a set of x positive integers and $\lambda_n(A)$ denotes the minimum eigenvalue of a $n \times n$ symmetric matrix A.

Consider a HMP in $\alpha \in \boldsymbol{\omega}_{\mathrm{N}}$ of an arbitrary integer degree $k > 1$ with $\chi(k) = \dfrac{(k+N-1)!}{k!(N-1)!}$, $0! = 1$, symmetric matrix coefficients, given by

$$\Pi(\alpha, k) = \sum_{|k|=k} \alpha_1^{k_1} \alpha_2^{k_2} \ldots \alpha_N^{k_N} P_{k_1 k_2 \ldots k_N} \in \mathbf{R}_n . \tag{1}$$

Since $|\alpha|^d = 1 \ \forall d = 0, 1, \ldots$, then the HMP (1) can be equivalently represented as a HMP of an arbitrary degree $d + k$ with $\chi(k+d)$ symmetric matrix coefficients

$$\begin{aligned} \Pi(\alpha,\ k) &= \Pi(\alpha, k+d) = |\alpha|^d \, \Pi(\alpha, k) = \\ &= \sum_{|k|=k+d} \alpha_1^{k_1} \alpha_2^{k_2} \ldots \alpha_N^{k_N} \overline{P}_{k_1 k_2 \ldots k_N} = \sum_{l=1}^{\chi(k+d)} \tilde{\alpha}_l \Pi_l \in \mathbf{R}_n , \end{aligned} \tag{2}$$

where $\tilde{\alpha}_l$ and Π_l, $l = 1, \ldots, \chi(k+d)$, denote the lexically ordered monomial $\alpha_1^{k_1} \alpha_2^{k_2} \ldots \alpha_N^{k_N}$ and the corresponding to it matrix coefficient $\overline{P}_{k_1 k_2 \ldots k_N}$, $|k| = k + d$, respectively. Let $k + d = 2\tau$, which makes possible to rewrite (2) as follows:

(3) $$\Pi(\alpha, k) = \Pi(\alpha, 2\tau) = \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \overline{\alpha}_i \overline{\alpha}_j \Pi_{ij},$$

where $\overline{\alpha}_i = \alpha_1^{\tau_1}\alpha_2^{\tau_2}...\alpha_N^{\tau_N}$, $|\tau| = \tau$, $i = 1, 2,..., \chi(\tau)$, denotes the i-th monomial of degree τ and $\tilde{\alpha}_l = \overline{\alpha}_i\overline{\alpha}_j$, $\Pi_l = \Pi_{ij}$ for some subscripts l, i and j.

Define the real uncertain Homogeneous Scalar Polynomial (HSP)

(4) $$f(\alpha, 2\tau, x) = x^{\mathrm{T}}\Pi(\alpha, k)x = x^{\mathrm{T}}\Pi(\alpha, 2\tau)x = \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \overline{\alpha}_i \overline{\alpha}_j \tilde{c}_{ij}(x), \quad x \in \mathbf{X}_n,$$

where $\tilde{c}_{ij}(x) = x^{\mathrm{T}}\Pi_{ij}x$ and the vector

$$\overline{\alpha}_v = (\overline{\alpha}_i)^{\mathrm{T}} \in \mathbf{R}^{\chi(\tau)}, \ \overline{\alpha}_i = \alpha_1^{\tau_1}\alpha_2^{\tau_2}...\alpha_N^{\tau_N}, \ |\tau| = \tau, \ i = 1, 2,..., \chi(\tau),$$

contains all monomials of degree τ . Then, (4) can be rewritten in a quadratic, with respect to $\overline{\alpha}_v$, compact matrix form as

(5) $$f(\alpha, 2\tau, x) = \overline{\alpha}_v^{\mathrm{T}} C(x)\overline{\alpha}_v, \ C(x) = [c_{ij}(x)] \in \mathbf{R}_{\chi(\tau)},$$

$$c_{ij}(x) = \pi_{ij}\tilde{c}_{ij}(x), \pi_{ij} = \begin{cases} 1, & i = j, \\ 0.5, & i \ne j. \end{cases}$$

The symmetric matrix $C(x)$ is said to be a Coefficient Matrix (CM) for the associated with (3) HSP $f(\alpha, 2\tau, x)$ in (5). It is desired to derive conditions under which the HMP in (3) is positive definite on the compact vector set $\boldsymbol{\omega}_N$, i.e., $\Pi(\alpha, k)$ contains only positive definite matrices, or equivalently, the strict scalar inequality

(6) $$f(\alpha, 2\tau, x) > 0 \ \ \forall\alpha \in \boldsymbol{\omega}_N, \forall x \in \mathbf{X}_n$$

holds. An well known asymptotically exact validity condition is given now.

Theorem 1 [14]. Let a given HMP in (1) be positive definite. There exists some sufficiently large integer d^*, such that for $d \ge d^*$, all $\chi(k+d)$ matrix coefficients of the HMP in (2) are positive definite.

Theorem 1 generalizes the famous Polya's theorem [4] for the case of matrix valued functions and it provides a systematic way to decide whether a given HMP is positive definite. Unfortunately, this result is very conservative with respect to sufficiency, due to the following obvious reasons. According to Theorem 1, the above stated problem has a solution if and only if for some appropriate d, all coefficients $\tilde{c}_{ij}(x) = x^{\mathrm{T}}P_{ij}x$ of the HSP $f(\alpha, \tau, x)$ are positive for all $x \in \mathbf{X}_n$. This means that some unduly large value for d^* may be required, in order to conclude positive definiteness of the HMP (1). As a result, serious computational problems may arise.

3. Relaxed analysis for HMPs

It is clear, that the scalar inequality in (6) may hold even if some coefficients are not positive. Therefore, the analysis procedure needs to be relaxed and made more effective.

The applied here relaxed analysis approach is described in details in [10]. It can be briefly summarized as follows.

Let $\alpha(s)$ denotes a vector with $s \geq 2$ arbitrarily selected entries from α. If $\mathbf{V}(s)$ is the set of $s \times 1$ vectors with entries representing an arbitrary non-descending sequence, then all possible systems of $s!0.5s(s–1)$ pairwise inequalities $\alpha_i \leq \alpha_j$, $\alpha_i, \alpha_j \in \alpha(s), i \neq j$, are described by the set of ordered vectors $\alpha_p(s) \in \mathbf{V}(s)$, $p = 1, \ldots, s!$.

For any $\alpha_p(s) \in \mathbf{V}(s)$, $\alpha_p(s) \in \mathbf{V}(s)$, $2 \leq s \leq N$, there exists a set of $\tilde{\mu}(s, \tau, N)$ inequalities between the monomials of degree 2τ, i.e., $\overline{\alpha}_i \overline{\alpha}_j \leq \overline{\alpha}_u \overline{\alpha}_v$, $i \leq j,\ u \leq v,\ ij \neq uv$. The total number of such inequalities, corresponding to all possible and compatible inequalities between the entries of vector $\alpha(s)$ is $\mu(s, \tau, N) = s!\tilde{\mu}(s, \tau, N)$.

For any $\alpha_p(s) \in \mathbf{V}(s)$, $p = 1, \ldots, s!$, these inequalities lead naturally to the matrix inequalities

$$(\overline{\alpha}_i \overline{\alpha}_j - \overline{\alpha}_u \overline{\alpha}_v) X_{ijuv,p} \leq 0\ \forall X_{ijuv,p} \geq 0,\ i \leq j,\ u \leq v,\ ij \neq uv.$$

Define, also, the associated with a given vector $\alpha_p(s) \in \mathbf{V}(s)$ two sets of HMPs:

$$\text{(7)} \qquad \tilde{\Pi}_p(\alpha, 2\tau) = \sum_{i,j \in \mathbf{N}_{\sigma(2\tau)},\, i \leq j} \overline{\alpha}_i \overline{\alpha}_j \Big(\sum_{\overline{\alpha}_i \overline{\alpha}_j \leq \overline{\alpha}_u \overline{\alpha}_v, u \leq v} X_{ijuv,p} - \sum_{\overline{\alpha}_i \overline{\alpha}_j > \overline{\alpha}_u \overline{\alpha}_v, u \leq v} X_{ijuv,p} \Big) =$$

$$= \sum_{i,j \in \mathbf{N}_{\sigma(2\tau)},\, i \leq j} \overline{\alpha}_i \overline{\alpha}_j (\tilde{\Pi}^{+}_{ij,p} + \tilde{\Pi}^{-}_{ij,p}) =$$

$$= \sum_{i,j=1,\, i \leq j}^{N} \overline{\alpha}_i \overline{\alpha}_j \tilde{\Pi}_{ij,p},\ p = 1, \ldots, s!;$$

$$\text{(8)} \qquad \Pi_p(\alpha, 2\tau) = \Pi(\alpha, 2\tau) + \tilde{\Pi}_p(\alpha, 2\tau) = \sum_{i,j=1, i \leq j}^{N} \overline{\alpha}_i \overline{\alpha}_j (\Pi_{ij} + \tilde{\Pi}_{ij,p}) =$$

$$= \sum_{i,j=1,\, i \leq j}^{N} \overline{\alpha}_i \overline{\alpha}_j \Pi_{ij,p},\ p = 1, \ldots, s!.$$

Having in mind the presentations (4) and (5), consider the p-th HSP

$$\tilde{f}_p(\alpha, 2\tau, x) + f(\alpha, 2\tau, x) = f_p(\alpha, 2\tau, x) =$$

$$= x^{\mathrm{T}} [\tilde{\Pi}_p(\alpha, 2\tau) + \Pi(\alpha, 2\tau)] x = x^{\mathrm{T}} \Pi_p(\alpha, 2\tau) x.$$

Denote $\Pi_{ij,p} = \tilde{\Pi}_{ij,p} + \Pi_{ij}$, $p = 1,\ 2, ..., \bar{p}$, and rewrite the above set of HSPs as:

(9) $$f_p(\alpha, 2\tau, x) = x^{\mathrm{T}} \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \bar{\alpha}_i \bar{\alpha}_j \Pi_{ij,p} x = \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \bar{\alpha}_i \bar{\alpha}_j \tilde{c}_{ij,p}(x) = \bar{\alpha}_v^{\mathrm{T}} C_p(x) \bar{\alpha}_v,$$
$$p = 1,\ 2, ..., \bar{p}.$$

Matrix $C_p(x) = [c(x)_{ij,p}] = \tilde{C}_p(x) + C(x) = [\tilde{c}_{ij,p}(x) + c_{ij}(x)]$, $p = 1,\ 2, ..., \bar{p}$, is the dependent on vector CM x, associated with the HMP $\Pi_p(\alpha, 2\tau)$, $p = 1, 2, ..., \bar{p}$ in (8).

Consider the HSPs

$$f_p(\alpha, 2\tau) = \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \bar{\alpha}_i \bar{\alpha}_j \tilde{c}_{ij,p},\ \ \tilde{c}_{ij,p} \le \tilde{c}(x)_{ij,p}\ \ \forall x \in \mathbf{X}_n,\ p = 1,\ 2, ..., \bar{p}.$$

It follows that

(10) $$f_p(\alpha, 2\tau) = \bar{\alpha}_v^{\mathrm{T}} C_p \bar{\alpha}_v \le f_p(\alpha, 2\tau, x)\ \ \forall \alpha \in \boldsymbol{\omega}_N, \forall x \in \mathbf{X}_n,\ p = 1,\ 2, ..., \bar{p},$$

where $C_p = [c_{ij,p}]$.

Theorem 2 [10]. The following statements are equivalent:

(i) a given HMP (1) is positive definite on the uncertainty set $\boldsymbol{\omega}_N$,

(ii) there exist appropriate numbers d, s = 0, 1,…, and $\bar{p}$ HMPs defined in (7), such that all associated with them CMs $C_p = [c_{ij,p}]$ in (10) are positive definite.

Theorem 2 states a new and more general condition in comparison with Theorem 1. It will be shown how this result can be used to improve the analysis procedure.

4. Robust stability analysis

Consider the uncertain linear system

(11) $$\dot{x} = A(\alpha)x, \quad A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i \in \mathbf{R}_n, \quad \alpha \in \boldsymbol{\omega}_N,$$

where all matrices A_i are fixed and Hurwitz (negative stable). The stability analysis problem for this class of uncertain systems is: determine necessary and sufficient conditions, under which the polytope $\mathbf{A} = \{A(\alpha) : \alpha \in \boldsymbol{\omega}_N\}$ contains only Hurwitz matrices.

An widely used approach to solve this problem consists in the determination of a positive definite on $\boldsymbol{\omega}_N$ HMP $\Pi(\alpha, t)$, such that the following exact condition be satisfied

(12) $$\Pi(\alpha, k) = -\{A^{\mathrm{T}}(\alpha)\Pi(\alpha, t) + \Pi(\alpha, t)A(\alpha)\} > 0,\ k = t + 1.$$

If the degree of polynomial dependence t is a positive integer, then matrix $\Pi(\alpha, t)$ is a parameter dependent one and the stability research approach is known as the Lyapunov's second (direct) method for robust stability analysis. It is important to underline, that the solution of this significant from both theoretical and practical point of view problem, consists actually in deriving conditions for positive definiteness of the HMP in (12). What's more – the efficiency of the derived in the literature approaches providing asymptotically exact conditions, depends crucially on them.

Consider the following three robust stability conditions for the uncertain system (11).

Theorem 3 [2, 3]. Let $d = s = 0$. Then the uncertain system is robustly stable if there exists a HMP $\Pi(\alpha, 1)$ of degree one, such that the single CM C, defined in (10) is positive definite.

Theorem 4 [7]. The uncertain system is stable if and only if there exist integers d and s, and a HMP $\Pi(\alpha, t)$, such that all $\chi(k+d)$ matrix coefficients of the HMP $\Pi(\alpha, k+d) = |\alpha|^d \; \Pi(\alpha, k)$ are positive definite.

Theorem 5 [11, 12]. The uncertain system is stable if and only if there exist integers d, t and s, a HMP $\Pi(\alpha, t)$ and s! HMPs $\tilde{\Pi}_p(\alpha, 2\tau)$ in (7), such that the defined in (10) CMs are all positive definite.

It is clear that Theorem 3 is a particular case of Theorem 5. All results due to Peres and Oliveira [6-9] are based on Theorem 1, which inevitably defines their main shortcoming – conservative sufficiency part of the robust stability condition and hence redundant and complicated computational procedures. It is also obvious, that Theorem 5 (based entirely on Theorem 2) generalizes Theorem 4, since if it's statement holds, than there always can be determined some appropriate parameters, such that the systems' stability can be easily concluded via Theorem 5, as well, but not vice versa. These facts will be illustrated by the considered here examples.

4. Experimental comparative analysis of uncertain models

The robust stability analysis of four real data uncertain dynamic models is considered in this section. A comparison regarding to some qualitative and quantitative indicators of the solutions, due to Theorems 3-5 is made. The matrix coefficients of the HMP $\Pi(\alpha, t)$ in (12) are given in the Appendix. They are obtained as a result of the solution of appropriately defined systems of LMIs by means of the LMI TOOLBOX [15].

Overhead crane control. The model of the plant to be controlled is taken from [13]. In this case, the crab mass is 1 t, the rope length is 10 m, and the load mass q varies in certain bounds, with a nominal value q_0 = 3 t. The model of the uncertain control system is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 9.81q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.981(q+1) & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10^{-3} \\ 0 \\ -10^{-4} \end{bmatrix} u = A(q)x + Bu; \;\; x = (x_1 \; x_2 \; x_3 \; x_4)^{\mathrm{T}},$$

$q \in [1, 10]$, where the first two entries of the state vector denote the velocity and position of the scrab, and the next couple denotes the angle and the angular velocity of the rope. The constraints regarding this system are given by the following inequalities

$$|u| \le 10^4\,\mathrm{N}, \;\; |x_0| \le (12\,\mathrm{m} \quad 1\,\mathrm{m/s} \quad 2\frac{\pi}{180^\circ} \quad 0.5\frac{\pi}{180^\circ.\mathrm{s}})^{\mathrm{T}}.$$

The control task is to drive the position of the load $y = (1 \quad 0 \quad 10 \quad 0)x$ from the initial state $|x_0| = (12\,\mathrm{m} \quad 0 \quad 0 \quad 0)^{\mathrm{T}}$ to the tolerance band крайно $|y| \le 2$ cm in minimum settling time. It is shown in [13], that the state control law $u = -Kx, \;\; K = [187.35 \quad 2869.3 \quad -26048 \quad 2455.3]$, meets the above requirements for all $q \in [1, 10]$. The task is to analyze the robust stability of the close loop system $\dot{x} = [A(q) + BK]x = A_C(q)x,$ and possibly extend the tolerance interval.

For the nominal load q = 3 t, the state matrix of the close loop system can be represented as $A_C(q) = A_C + qA_q; \;\; q \in [q_L, q_U], \;\; q_L \le -2, \;\; q_U \ge 7$.

The two matrices in (11) are $A_1 = A_C + q_U A_q, \;\; A_2 = A_C - q_L A_q$. Let $q_L = -2, \;\; q_U = 7.5.$ For the robust stability analysis a parameter dependent CLF $x^{\mathrm{T}}\Pi(\alpha, 3)x$ of degree three with four matrix coefficients $\Pi_{30}, \Pi_{03}, \Pi_{21}, \Pi_{12}$, is used. The minimal eigenvalues of the five coefficients of the HMP (12) are denoted accordingly $\lambda_{xy}, \, x + y = 4,$ and are computed as: $\lambda_{40} = 0.014, \lambda_{04} = 0.05, \lambda_{22} = -3.32 \times 10^{-4}, \lambda_{31} = 0.003, \lambda_{13} = 0.0046$.

According to Theorem 4, the considered CLF is not a Valid Lyapunov Function (VLF) for the system for $d = 0$.

The two possible cases are considered next:

1. $\alpha_1 \ge \alpha_2 \Rightarrow (\alpha_1^2\alpha_2^2 - \alpha_1^3\alpha_2)\Pi_{31} \le 0$, since $\Pi_{31} > 0$, which leads to the matrix inequality $\Pi_{22,1} = \Pi_{22} + \Pi_{31} > 0, \lambda_{22,1} = 0.04$. Then, the CM C_1 can be chosen as a positive definite diagonal matrix, i.e., the system is robustly stable according to Theorem 5.

2. $\alpha_1 < \alpha_2 \Rightarrow (\alpha_1^2\alpha_2^2 - \alpha_1\alpha_2^3)\Pi_{13} \le 0.$ Then,

$$\Pi_{22,2} = \Pi_{22} + \Pi_{13} > 0, \lambda_{22,2} = 0.046$$

and again the CM C_2 can be chosen as a positive definite diagonal matrix. The close loop uncertain system remains robustly stable for the extended admissible parameter set $[1, 10.5]$.

Electromechanical system. Such systems are constructed with components that manifest a combination of inertial, compliant, and dissipative effects. In practice, engineers typically model the dynamic response of these with finite element codes or lumped physical models. Each of these ultimately leads to a set of constant coefficient, ordinary differential equations. A drive model containing three flywheels, two dashpots and two springs is considered in [5] and given by the system equation $J\ddot{\theta} + B_S\dot{\theta} + K_S\theta = T$, $\theta = (\theta_i) \in \mathbf{R}^3$, $T = (t_i) \in \mathbf{R}^3$, where θ is a vector of the angular displacements, $J = [j_i] \in \mathbf{R}_3$ is a diagonal damping matrix. The coefficient matrices B_S, K_S and the input vector u are:

$$B_S = \begin{bmatrix} b_1 & -b_1 & 0 \\ -b_1 & b_1 + b_2 & -b_2 \\ 0 & -b_2 & b_2 \end{bmatrix}, \quad K_S = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}, \quad u = \begin{pmatrix} T_{in} \\ 0 \\ 0 \end{pmatrix}.$$

Real world electromechanical systems have parametric uncertainty (in some elements of inertia, damping and stiffness matrices), that must be considered during the design. When constructing the model, matrix B_S is taken as an uncertain one. Denoting the state vector $x = (\theta_1\ \theta_2\ \theta_3\ \omega_1\ \omega_2\ \omega_3)^{\mathrm{T}}$, where $\omega_1, \omega_2, \omega_3$ are the angular velocities of the respective wheels, one gets the usual state space model $\dot{x} = Ax + Bu$, where the state and control matrices are:

$$A = \begin{bmatrix} 0 & I \\ -J^{-1}K_S & -J^{-1}B_S \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}.$$

The uncertain parameters in this description are the elements of matrix K_S, which are considered as constant, but not fixed real numbers. It is assumed also, that $k_i = k_{i0} + \delta k_i$, $\delta k_i \in [-L, L]$, $i = 1, 2$.

The following values for the fixed system's parameters are borrowed from [5]:

$$j_1 = j_2 = j_3 = 1, \quad k_{10} = 10, \quad k_{20} = 20, \quad b_1 = 0.1, \quad b_2 = 0.2,$$

which makes possible to get the state matrix as

$$A = A_0 + \delta k_1 A_{k1} + \delta k_2 A_{k2},$$

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & -0.1 & 0.1 & 0 \\ 10 & -30 & 20 & 0.1 & -0.3 & 0.2 \\ 0 & 20 & -20 & 0 & 0.2 & -0.2 \end{bmatrix}, \quad 0 \in \sigma(A).$$

The output consists of the angular displacements of the first and second flywheel, i. e.,

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x.$$

Since the open loop system ($\delta k_1 = \delta k_2 = 0$) is not asymptotically stable, a stabilizing output feedback $u = Ky$ is firstly applied, where K is a 2×2 negative diagonal matrix with entries 40 and –9. The close loop system is described by the uncertain linear model

$$\dot{x} = A_C(\delta k)x, \quad A_C(\delta k) = A_{C0} + \delta k_1 A_{k2} + \delta k_2 A_{k2}, \quad A_{C0} = A_0 + BKC .$$

One can easily verify that the nominal close loop state matrix A_{C0} is a Hurwitz stable one. The respective four matrices in the uncertain model (11) are computed as follows:

$$A_1 = A_{C0} + L(A_{k1} + A_{k2}),\ A_2 = A_{C0} + L(A_{k1} - A_{k2}),$$
$$A_3 = A_{C0} + L(-A_{k1} + A_{k2}),\ A_4 = A_{C0} - L(A_{k1} + A_{k2}) .$$

Let L = 1.66, i.e., $\delta k_i \in [-1.66, 1.66], \quad i = 1, 2$. The robust stability analysis is performed via a linear parameter dependent candidate for a Lyapunov function (CLF) $x^{\mathrm{T}}\Pi(\alpha,\ 1)x$.

Only four, from all the six matrix coefficients of the HMP (12) of degree two are positive definite. According to Theorem 4, the considered CLF is not a valid Lyapunov function (VLF) for the system for d = 0. The single CM C is not positive definite and from Theorem 3 it follows, that the system’s stability can not be verified for this CLF.

The following results are obtained when Theorem 5 is applied. Let $\alpha(2) = (\alpha_2 \ \ \alpha_4)^{\mathrm{T}}$ be the chosen vector. The following two possible compatible cases are considered:

1. $$\alpha_2 \geq \alpha_4 \Rightarrow (\alpha_2\alpha_4 - \alpha_2^2)X_{2422,1} \leq 0,$$
$$X_{2422,1} = \Pi_{22} - \lambda_n(\Pi_{22})I \geq 0 \Rightarrow C_1 > 0;$$

2. $$\alpha_2 < \alpha_4 \Rightarrow (\alpha_2^2 - \alpha_2\alpha_4)X_{2224,2} \leq 0,$$
$$X_{2224,2} = \Pi_{24} + X_{2444,2} - \lambda_n(\Pi_{24} + X_{2444,2})I \geq 0, \text{ and}$$
$$\alpha_2 < \alpha_4 \Rightarrow (\alpha_2\alpha_4 - \alpha_4^2)X_{2444,2} \leq 0, \quad X_{2444,2} = \Pi_{44} - \lambda_n(\Pi_{44})I \geq 0 \Rightarrow C_2 > 0.$$

The matrix coefficients of the HMPs $\Pi_p(\alpha, 2),\ p = 1, 2$, are computed as follows

$$\Pi_{11,1} = \Pi_{11}, \Pi_{22,1} = \lambda_{22}I, \Pi_{33,1} = \Pi_{33}, \Pi_{44,1} = \Pi_{44},$$
$$\Pi_{12,1} = \Pi_{12}, \Pi_{13,1} = \Pi_{13}, \Pi_{14,1} = \Pi_{14}, \Pi_{24,1} = \Pi_{24} + X_{2422,1}, \Pi_{23,1} = \Pi_{23}, \Pi_{34,1} = \Pi_{34} .$$

The usage of one single parametric matrix is enough to get the CM C_1 positive definite, when $\alpha_2 \geq \alpha_4$. If $\alpha_2 < \alpha_4$, then simple computations show, that:

$$\Pi_{11,2} = \Pi_{11}, \Pi_{22,2} = \Pi_{22} + X_{2224,2}, \Pi_{33,2} = \Pi_{33}, \Pi_{44,2} = \lambda_{44}I ,$$
$$\Pi_{12,2} = \Pi_{12}, \Pi_{13,2} = \Pi_{13}, \Pi_{14,2} = \Pi_{14},$$
$$\Pi_{24,1} = \Pi_{24} + X_{2422,2} - X_{2224,2} = \lambda_4(\Pi_{24} + X_{2444,2})I,$$
$$\Pi_{23,2} = \Pi_{23}, \Pi_{34,2} = \Pi_{34} .$$

This choice for the parametric matrices clearly illustrates the advantages of the approach, since the minimal eigenvalues of the modified matrix coefficients $\Pi_{22,2} = \Pi_{22} + X_{2224,2} \geq \Pi_{22}$ and $\Pi_{24,1} = \lambda_4(\Pi_{24} + X_{2444,2})I \geq \lambda_4(\Pi_{24})I$, are increased. The CM C_2 is proven to be positive definite, as well. Thus, the considered function is a VLF (for d = 0) for the robustly stable uncertain system.

A value for $L = 0.3016$, which guarantees stability, is obtained in [5]. From Theorem 5, it follows, that the system retains it's stability even for L = 1.66, which is more than five times extension of the admissible set for the uncertain parameters $\delta k_i \in [-L, L], \quad i = 1, 2$.

Induction motor. The control object is described by the system of nonlinear differential equations [1]:

$$\dot{x} = f(x) + gu, \quad x = (\omega \quad \psi_{R\alpha} \quad \psi_{R\beta} \quad i_{S\alpha} \quad i_{S\beta})^{\mathrm{T}}, \quad u = (u_{S\alpha} \quad u_{S\beta})^{\mathrm{T}},$$

where ω denotes the induction machine speed, $\psi_{R\alpha}$, $\psi_{R\beta}$, $i_{S\alpha}$, $i_{S\beta}$ and $u_{S\alpha}$, $u_{S\beta}$ are the transformed rotor flux couple, stator current couple and stator voltage input couple, respectively.

Resistance (R) and mutual inductance (L) may vary due to rotor heating. By means of an identification procedure of different physical parameters at different setting points, it has been established, that the usual percentage variations in these parameters is within the following intervals $\delta R \in [-50, 50]$ и $\delta L \in [-20, 20]$. The system is highly nonlinear, but application of a input-output linearization procedure and some physical parameters' modification, results in the standard state space linear model

$$\dot{x} = (A_n + \delta R A_R + \delta L A_L)x = A(\delta)x, \quad A(\delta) \in \mathbf{R}_4,$$

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \alpha_n \\ -\rho_{\omega_1}\rho_{\omega_2} & \rho_{\omega_1} + \rho_{\omega_2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_n & 0 & -\rho_{|\psi_1|^2}\rho_{|\psi_2|^2} & -\rho_{|\psi_1|^2} - \rho_{|\psi_2|^2} \end{bmatrix},$$

$$A_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma_n - \alpha_n & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2\alpha_n(-\gamma_n - \alpha_n + \alpha_n\beta_n\mu_{SRn}) & -\gamma_n - 3\alpha_n \end{bmatrix},$$

$$A_L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \rho_{\omega_1}\rho_{\omega_2} & -\rho_{\omega_1} - \rho_{\omega_2} & 0 & 0 \\ 0 & 0 & 2\alpha_n & 0 \\ 0 & 0 & -\rho_{|\psi_1|^2}\rho_{|\psi_2|^2} - 2\alpha_n^2(\beta_n\mu_{SRn} + 2) & -2\alpha_n - \rho_{|\psi_1|^2} - \rho_{|\psi_2|^2} \end{bmatrix},$$

$$\alpha_n = R_{Rn} / L_{Rn}, \beta_n = \mu_{SRn} / \sigma_n L_{Sn} L_{Rn}, \quad \sigma_n = (1 - \mu_{SRn}) / L_{Sn} L_{Rn}.$$

The following fixed parameter values are used in [12]:

$L_{Sn} = 471.8$ mH – stator inductance;

$\mu_{SRn} = 441.5$ mH – mutual inductance;

$L_{Rn} = 471.8$ mH – rotor inductance;

$R_{Sn} = 9.65\,\Omega$ – stator resistance;

$R_{Rn} = 4.305\,\Omega$ – rotor resistance;

$\rho_{\omega_1} = -30,\ \rho_{\omega_2} = -300,\ \rho_{|\psi_1|^2} = \rho_{|\psi_2|^2} = -310.$

The main purpose of the robust stability analysis is to find out, whether the system is stable and if it is possible to extend the admissible variation set $[-50, 50] \cap [-20, 20]$. Let $\delta R \in [-r, r]$, r = 70% and $\delta L \in [-l, l]$, l = 30%. This corresponds to a 40% and 50% increase in the interval bounds for R и L, respectively.

The linear model can be put in the form (11), where the matrix polytope is completely described by its four Hurwitz stable vertices

$$A_1 = A_n + 0.7A_R + 0.3A_L,\ A_2 = A_n + 0.7A_R - 0.3A_l,\ A_3 =$$
$$= A_n - 0.7A_R + 0.3A_L,\ A_4 = A_n - 0.7A_R - 0.3A_l.$$

The robust stability analysis is performed via affine parameter dependent CLF. The minimal eigenvalues of the matrix coefficients of the $\Pi(\alpha, 2)$ in (12) are computed as follows: $\lambda_{11} = \lambda_{22} = 1$, $\lambda_{33} = \lambda_{44} = 0.3$, $\lambda_{12} = 1.992$, $\lambda_{13} = 1.986$, $\lambda_{14} = 0.4094$, $\lambda_{23} = -0.1383$, $\lambda_{24} = 1.9775$, $\lambda_{34} = -2.3572$. Therefore, matrices Π_{23}, Π_{34} are not positive definite and the considered function is not a VLF for the system, in accordance with Theorem 4, for d = 0. The respective CM C is not a positive definite one, which means that according to Theorem 3 systems' stability can not be verified via this CLF.

A based on Theorem 5 solution is suggested. Since some matrix coefficients are positive definite, a simplified analysis procedure is applied, which investigates the positive definiteness of the following HMP

$$\Pi_L(\alpha, 2) = \alpha_1^2\Pi_{11} + \alpha_2^2\Pi_{22} + \alpha_3^2\Pi_{33} + \alpha_4^2\Pi_{44} + \alpha_2\alpha_3\Pi_{23} + \alpha_2\alpha_4\Pi_{24} + \alpha_3\alpha_4\Pi_{34} \le$$
$$\le \Pi(\alpha, 2)_{(12)} \quad \forall \alpha \in \mathbf{\Gamma}.$$

If $\alpha \equiv (\alpha_1\ 0\ 0\ 0)^{\mathrm{T}}$, i.e., $\alpha_1 = 1$, then the system is stable since $\Pi_{11} > 0$.

Let $0 < \alpha_2 + \alpha_3 + \alpha_4 \le 1$. Then the following matrix inequality holds:

$$\Pi_{L1}(\alpha,\ 2) = \alpha_2^2\Pi_{22} + \alpha_3^2\Pi_{33} + \alpha_4^2\Pi_{44} + \alpha_2\alpha_3\Pi_{23} + \alpha_2\alpha_4\Pi_{24} + \alpha_3\alpha_4\Pi_{34} \le$$
$$\le \Pi_L(\alpha, 2) \le \Pi(\alpha, 2)_{(12)}$$

for $\forall \alpha \in \mathbf{\Gamma}$, $0 < \alpha_2 + \alpha_3 + \alpha_4 \le 1$. If the HMP of degree four

$$\Pi(\alpha, 4) = (\alpha_2 + \alpha_3 + \alpha_4)^2\,\Pi_{L1}(\alpha, 2),\ 0 < \alpha_2 + \alpha_3 + \alpha_4 \le 1,$$

is positive definite, then it follows that $\Pi(\alpha, 2)_{(12)} > 0\ \forall \alpha \in \mathbf{\Gamma}$, or, equivalently, the uncertain system is robustly stable via the considered CLF. The HMP $\Pi(\alpha,4)$ is given by

$$\Pi(\alpha,4)=\alpha_2^4\Pi_{400}+\alpha_3^4\Pi_{040}+\alpha_4^4\Pi_{004}+\alpha_2^3\alpha_3\Pi_{310}+\alpha_2^2\alpha_3^2\Pi_{220}+\alpha_2\alpha_3^3\Pi_{130}+$$
$$+\alpha_2^3\alpha_4\Pi_{301}+\alpha_2^2\alpha_4^2\Pi_{202}+\alpha_2\alpha_4^3\Pi_{103}+\alpha_3^3\alpha_4\Pi_{031}+\alpha_3^2\alpha_4^2\Pi_{022}+\alpha_3\alpha_4^3\Pi_{013}+$$
$$+\alpha_2^2\alpha_3\alpha_4\Pi_{211}+\alpha_2\alpha_3^2\alpha_4\Pi_{121}+\alpha_2\alpha_3\alpha_4^2\Pi_{112}\,.$$

All matrix coefficients, except for Π_{022} and Π_{031} are positive definite. Let λ_{abc}, $a+b+c=4$, denotes the minimal eigenvalue of the matrix coefficient Π_{abc}. Then, the corresponding to the HMP ХМП $\Pi(\alpha,4)$ CM is computed as

$$2C=\begin{bmatrix} 2\lambda_{400} & 0 & 0 & \lambda_{310} & \lambda_{301} & \lambda_{211} \\ 0 & 2\lambda_{040} & 0 & \lambda_{130} & \lambda_{121} & \lambda_{031} \\ 0 & 0 & 2\lambda_{004} & \lambda_{112} & \lambda_{103} & \lambda_{013} \\ \lambda_{310} & \lambda_{130} & \lambda_{112} & 2\lambda_{220} & 0 & 0 \\ \lambda_{301} & \lambda_{121} & \lambda_{103} & 0 & 2\lambda_{202} & 0 \\ \lambda_{211} & \lambda_{031} & \lambda_{013} & 0 & 0 & 2\lambda_{022} \end{bmatrix}.$$

Since the diagonal entry λ_{022} is negative, the CM is not positive definite. The problem's solution by means of Theorem 5 is described next.

Let the selected vector be $\alpha(2)=(\alpha_3\ \alpha_4)^{\mathrm{T}}$ .The two possible cases are:

$p=1:\quad \alpha_3\le\alpha_4\Rightarrow(\alpha_3^2\alpha_4^2-\alpha_3\alpha_4^3)\Pi_{013}\le 0,\ \text{since}\ \Pi_{013}>0$.

This produces the following changes in the matrix coefficients of the HMP $\Pi_1(\alpha,4)$:

$\Pi_{022,1}=\Pi_{022}+\Pi_{013}>0,\quad \lambda_{022,1}=0.2234, \Pi_{013,1}=0$. The construction of the CM $C_1>0$ is done in accordance with the requirements of Theorem 5.

$p=2:\quad \alpha_3>\alpha_4\Rightarrow(\alpha_3^2\alpha_4^2-\alpha_3^3\alpha_4)(\Pi_{031}-\lambda_{031}I)\le 0,\ \text{because}\ \Pi_{031}-\lambda_{031}I\ge 0$.

The new matrix coefficients of the HMP $\Pi_2(\alpha,4)$ are:

$$\Pi_{022,2}=\Pi_{022}+\Pi_{031}-\lambda_{031}I\ ,\ \ \Pi_{031,2}=\lambda_{031}I\ .$$

The CM C_2 is also a positive definite one. Therefore, according to Theorem 5 the system is robustly stable and what's more – the admissible parameter set has been significantly extended, in comparison with the one obtained in [1].

Electric power system. A linearised model of a power system, comprised of k subsystems (generators) is considered

$$\Delta\dot{\omega}_i=-R_i^{-1}\Delta\omega_i+\sum_{j=1}^{N}Y_{ij}\delta_i,\quad Y_{ij}=Y_{ji},\quad i=1,2,\ldots,k,$$

$$\dot{\delta}_i=\Delta\omega_i,\quad i=1,2,\ldots,k\ ,$$

where δ_i, $i=1,2,\ldots,k$, denote the deviation in the angle of the i-th rotor from the nominal value; Y_{ij} is the transfer conductance between subsystems i и j and Y_{ii} is the self conductance of the i-th subsystem. It is assumed that all coefficients in the $2k$ differential equations do not vary in time. Denoting the vector

$$x=(x_1^{\mathrm{T}}\ x_2^{\mathrm{T}}...x_i^{\mathrm{T}}...x_N^{\mathrm{T}})^{\mathrm{T}}\in\mathbf{R}^N,\quad N=2k,\quad x_i=(\Delta\omega_i\ \ \delta_i)^{\mathrm{T}}\in\mathbf{R}^2,$$

the system's dynamics description can be put in a compact matrix form, where the state matrix has the following block structure:

$$A=\begin{bmatrix}A_{11} & A_{12} & \dots & A_{1k}\\ A_{12} & A_{22} & \dots & A_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ A_{1k} & A_{2k} & \dots & A_{kk}\end{bmatrix}\in\mathbf{R}_N;\quad A_{ii}=\begin{bmatrix}-R_i^{-1} & Y_{ii}\\ 1 & 0\end{bmatrix}, A_{ij}=\begin{bmatrix}0 & Y_{ij}\\ 0 & 0\end{bmatrix}\in\mathbf{R}_2.$$

The power systems belong to the class of decentralized control systems. The most important practical problem is to answer the question whether the system retains its stability when some or all interconnections fail partially or completely. It is justified to use the model [16]:

$$\dot{x}=(\tilde{A}_0+\sum_{t=1}^{s}\varepsilon_t\tilde{A}_t)x,\ s=0.5k(k-1),\ \varepsilon_t\in[0,1],\ \tilde{A}_0=\mathrm{bl.diag}(A_{ii}),\ i=1,...,k,$$

$$\tilde{A}_t=\begin{bmatrix}0 & \dots & A_{ij} & \dots & 0\\ \vdots & \ddots & \vdots & \iddots & \vdots\\ A_{ij} & \dots & 0 & \dots & \vdots\\ \vdots & \iddots & \vdots & \ddots & \vdots\\ 0 & \dots & \dots & \dots & 0\end{bmatrix},\ t=1,...,s.$$

(i) Let k = 2, n = 4, s = 1. For the typical values of the parameters $R_{ii}=0.01, i=1,2, Y_{11}=-2.2, Y_{22}=-2.6, Y_{12}=1,$ the three matrices are computed as

$$A_{11}=\begin{bmatrix}-100 & -2.2\\ 1 & 0\end{bmatrix},\ A_{22}=\begin{bmatrix}-100 & -2.6\\ 1 & 0\end{bmatrix},\ A_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

It is desired to find out if the system is robustly stable for all possible values of the uncertain parameter ε_1. The corresponding equivalent representation (11) is easily obtained, where $A_1=\tilde{A}_0, A_2=\tilde{A}_0+\tilde{A}_1$. For analysis purposes a parameter dependent affine CLF is used, where

$$P_1=\begin{bmatrix}0.0145 & 0.4545 & -0.0005 & -0.0021\\ 0.4545 & 44.9865 & 0.0018 & -0.2178\\ -0.0005 & 0.0018 & 0.0138 & 0.3846\\ -0.0021 & -0.2178 & 0.3846 & 37.6975\end{bmatrix}, P_2=\begin{bmatrix}2500.8 & 5.5 & -0.025 & -2.5\\ 5.5 & 2.01 & -2.5 & -1.45\\ -0.025 & -2.5 & 250.09 & 6.5\\ -2.5 & -1.45 & 6.5 & 25.9\end{bmatrix}.$$

Matix Π_{12} of the HMP $\Pi(\alpha,2)$ in (12) is not positive definite ($\lambda_n(\Pi_{12})=-0.1218$)); therefore, according to Theorem 4, the CLF is not a VLF for the system for d = 0. The single CM is computed as follows:

$$C_0=\begin{bmatrix}\lambda_4(\Pi_{11}) & 0.5\lambda_4(\Pi_{12})\\ 0.5\lambda_4(\Pi_{12}) & \lambda_4(\Pi_{22})\end{bmatrix}=\begin{bmatrix}1.036 & -0.0609\\ -0.0609 & 0.000952\end{bmatrix},\quad \lambda_2(C_0)=-0.0026,$$

i.e., it is not positive definite and robust stability can not be verified by means of Theorem 3, as well. Consider the cases:

a) $\alpha_1 \geq \alpha_2 \Rightarrow (\alpha_1\alpha_2 - \alpha_1^2)X_1 \leq 0,\ X_1 = \Pi_{11} - \lambda_4(\Pi_{11})I \geq 0$, which makes possible the determination of the coefficients of the HMP in (8) for $p = 1$ as

$$\Pi_{11,1} = \lambda_4(\Pi_{11})I,\ \Pi_{12,1} = \Pi_{12} + X_1 > 0,\ \Pi_{22,1} = \Pi_{22},\ \lambda_4(\Pi_{12,1}) = 0.809,$$

or the considered CFL is a VLF for the system for $\forall \alpha_1 \geq \alpha_2$.

b) $\alpha_1 < \alpha_2 \Rightarrow (\alpha_1\alpha_2 - \alpha_2^2)X_2 \leq 0,\ X_2 = \Pi_{22} - \lambda_4(\Pi_{22})I \geq 0$, which for $p = 2$ leads to

$$\Pi_{11,2} = \Pi_{11},\ \Pi_{12,2} = \Pi_{12} + X_2 > 0,\ \Pi_{22,2} = \lambda_4(\Pi_{22}),\ \lambda_4(\Pi_{12,2}) = 0.507,$$

or the considered CLF is a VLF for $\forall \alpha_1 < \alpha_2$, as well. This guarantees the robust properties of the system for $d = 0$, in accordance with Theorem 5.

(ii) Let $k = 3$, $n = 6$, $s = 3$. In this case, the state matrix has the structure

$$\tilde{A}_0 = \text{bl.diag}(A_{ii}),\ i = 1, 2, 3;\ \tilde{A}_1 = \begin{bmatrix} 0 & A_{12} & 0 \\ A_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{13} & 0 & 0 \end{bmatrix},\quad \tilde{A}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{23} & 0 \end{bmatrix},$$

and the following typical values have been assumed:

$$R_{ii} = 0.01, i = 1, 2, 3;\ Y_{11} = -2.2,\ Y_{22} = -2.2,\ Y_{33} = -2.3, Y_{12} = 1, Y_{13} = 1.3, Y_{23} = 1.2.$$

The vertex matrices in (11) are computed from the equalities

$$A_1 = \tilde{A}_0,\ A_2 = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3,\ A_3 = \tilde{A}_0 + \tilde{A}_2,\ A_4 = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_3,$$

$$A_5 = \tilde{A}_0 + \tilde{A}_1, A_6 = \tilde{A}_0 + \tilde{A}_2 + \tilde{A}_3,\ A_7 = \tilde{A}_0 + \tilde{A}_2,\ A_8 = \tilde{A}_0 + \tilde{A}_3.$$

Robust stability of the system is studied via an affine CLF.

The HMP $\Pi(\alpha, 2)$ in (12) has 36 matrix coefficients, two of which (Π_{12} and Π_{13}) are not positive definite. According to Theorem 4, the system is not stable for $d = 0$. The single CM is

$$2C_0 = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} \in \mathbf{R}_8;\ C_{11} = \begin{bmatrix} 2\lambda_6(\Pi_{11}) & \lambda_6(\Pi_{12}) & \lambda_6(\Pi_{13}) \\ \lambda_6(\Pi_{12}) & 2\lambda_6(\Pi_{22}) & 0 \\ \lambda_6(\Pi_{13}) & 0 & 2\lambda_6(\Pi_{33}) \end{bmatrix},$$

$$D = \text{bl.diag}(\lambda_6(\Pi_{ii})),\ i = 4, ..., 8,$$

where $D > 0$ and the respective minimal eigenvalues are computed as follows:

$$\lambda_6(\Pi_{11}) = 0.308, \lambda_6(\Pi_{22}) = 0.0043, \lambda_6(\Pi_{33}) = 0.00066,$$

$$\lambda_6(\Pi_{12}) = -0.0284, \lambda_6(\Pi_{13}) = -0.00002114.$$

According to Theorem 3, the system is not robustly stable, since $\lambda_3(C_{11}) = -0.001152$, i.e., the CM C_0 is not positive definite.

Let $\alpha(2) = (\alpha_1 \;\; \alpha_2)^{\mathrm{T}}$ be the chosen vector and consider the two possible cases:

a) $\alpha_1 \geq \alpha_2 \Rightarrow (\alpha_1\alpha_2 - \alpha_1^2)X_1 \leq 0, X_1 = \Pi_{11} - \lambda_6(\Pi_{11})I \geq 0$, which naturally leads to

$$\Pi_{11,1} = \lambda_6(\Pi_{11})I, \Pi_{12,1} = \Pi_{12} + X_1 > 0, \Pi_{22,1} = \Pi_{22},$$
$$\lambda_6(\Pi_{12,1}) = 0.0001, \Pi_{22,1} = \Pi_{22}, \Pi_{13,1} = \Pi_{13}, \Pi_{33,1} = \Pi_{33}.$$

b) $\alpha_1 < \alpha_2 \Rightarrow (\alpha_1\alpha_2 - \alpha_2^2)X_2 \leq 0, X_2 = \Pi_{22} - \lambda_6(\Pi_{22})I \geq 0$, which naturally leads to

$$\Pi_{11,2} = \Pi_{11}, \Pi_{12,2} = \Pi_{12} + X_2 > 0, \Pi_{22,2} = \lambda_4(\Pi_{22})I,$$
$$\lambda_6(\Pi_{12,2}) = 0.00021, \Pi_{22,2} = \lambda_6(\Pi_{22})I, \Pi_{13,2} = \Pi_{13}, \Pi_{33,2} = \Pi_{33}.$$

The two CMs in (10) can be chosen in accordance with Theorem 5 as $C_1 = C_2$ and then

$$C_p = \begin{bmatrix} C_{11,p} & 0 \\ 0 & D \end{bmatrix}, \quad C_{11,p} = \begin{bmatrix} 0.616 & 0 & -0.000021 \\ 0 & 0.0047 & 0 \\ -0.000021 & 0 & 0.00132 \end{bmatrix},$$
$$\lambda_3(C_{11,p}) = 0.00264, \; p = 1, 2,$$

i.e., the considered CLF is a VLF for the robustly stable system.

5. Comparative analysis of the experimental data

Table 1 contains information, concerning the efficiency of the robust stability conditions, provided by Theorems 3, 4 and 5.

Table 1. Stability conditions validity

EXAMPLE	*n*	*N*	Degree of CLF	Theorem 3	Theorem 4	Theorem 5
Overhead crane	4	2	3	not valid	VLF (*d*= 5)	VLF (*d*=0)
Electromechanical system	6	4	1	not VLF	VLF (*d*=6)	VLF (*d*=0)
Induction motor	4	4	1	not VLF	VLF (*d*=9)	VLF (*d*=2)
Power system	4	2	1	not VLF	VLF (*d*=7)	VLF (*d*=0)
Power system	6	8	1	not VLF	VLF (*d*=5)	VLF (*d*=0)

In all considered cases Theorem 3 is either inapplicable, or system's stability can not be verified via it. As far as Theorem 4 is regarded, one gathers the impression, that an obligatory increase in parameter *d* is required to solve the problem. It's just this fact which defines the computational complexity of the conditions given by Theorems 4 and 5, e.g., for the analysis of the induction motor, Theorem 4 needs a value *d* = 6, in order to conclude it's robust stability. This parameter value requires the solution of 4004 linear LMIs, and what's more – the respective asymptotically exact condition requires all matrix coefficients to be positive definite. The requirements towards the computational procedure imposed by Theorem 4 are very hard for the analysis of the electromechanical system (1980 LMIs) and the power system (1716 LMIs for case $N = 8$), as well. The number L of the LMIs needed to be solved, together with the number p of the involved matrix parameters are taken as quantitative measures of the computational procedure.

Table 2. Computational complexity parameters

Example	Theorem 4	Theorem 5
Overhead crane	$L = 92$, $p = 2$	$L = 7$, $p = 4$
Electromechanical system	$L = 1980$, $p = 2$	$L = 17$, $p = 7$
Induction motor	$L = 4004$, $p = 4$	$L = 41$, $p = 6$
Power system	$L = 10$, $p = 2$	$L = 3$, $p = 4$
Power system	$L = 1716$, $p = 8$	$L = 36$, $p = 10$

It can be easily seen, that with an insignificant increase in the number of parameters p , a considerable reduction in the number L of LMIs is obtained. The numbers $\Delta L = \frac{L_4}{L_5} > 1$ and $\Delta p = \frac{p_5}{p_4} > 1$ are generalised indicative parameters for the computational complexity required by the respective asymptotically exact conditions of Theorems 4 and 5. These numbers illustrate in a most convincing way the advantages of the new approach.

Table 3. Computational complexity – comparative analysis

Examples	ΔL	Δp
Overhead crane	13.143	2.0
Electromechanical system	116.5	3.5
Induction motor	97.585	1.5
Power system	3.333	2.0
Power system	47.666	1.25

These results clearly show that taking into account the inequalities between the entries of the uncertain parameter vector leads to a less conservative stability condition and hence, to a considerable decrease in the computational complexity, required by Theorem 5.

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Appendix

Overhead crane

Π30 =
1.0e+002 *
  0.00047018763952   0.00054336634955   –0.00028904891464
0.00498306863875
  0.00054336634955   0.02746078772649   –0.00604924571726
0.25418115719357
  –0.00028904891464 –0.00604924571726   2.77594243980907
–0.00950269586388
  0.00498306863875   0.25418115719357   –0.00950269586388
2.56128912943461
Π03 =
  0.07549530274587   0.06597226252721   –0.00724298891182
0.49873188838639
  0.06597226252721   0.96838137786520   –0.63049175685797
7.32356279296653
  –0.00724298891182 –0.63049175685797 81.24555889879822
–1.01342205424938
  0.49873188838639   7.32356279296653   –1.01342205424935

75.05602712689547

Π21 =
1.0e+002 *

0.00169532830650	0.00174645532438	–0.00065052771839	0.01495345616137
0.00174645532438	0.06460538923164	–0.01840340900310	0.58159794231680
–0.00065052771839	–0.01840340900310	6.36434046860613	–0.02913961227026
0.01495345616137	0.58159794231680	–0.02913961227025	5.87313853013817

Π12 =
1.0e+002 *

0.00198009369444	0.00186281160010	–0.00043390869287	0.01495770640648
0.00186281160010	0.04682841528380	–0.01865908085442	0.40065241305290
–0.00043390869287	–0.01865908085442	4.40085361778504	–0.02977113694887
0.01495770640648	0.40065241305290	–0.02977113694887	4.06240967197252

Electromechanical system

P1 =

158.5124	–32.4501	–2.8721	–0.1971	–3.0868	2.4999
–32.4501	26.8689	–17.0481	3.1861	1.8617	0.4062
–2.8721	–17.0481	17.5968	–2.5311	–0.8240	–0.3375
–0.1971	3.1861	–2.5311	3.0743	0.0425	–0.0816
–3.0868	1.8617	–0.8240	0.0425	0.5134	–0.0192
2.4999	0.4062	–0.3375	–0.0816	–0.0192	0.7974

P2 =

157.2995	–35.2147	–0.3470	–0.1950	–3.0776	1.9766
–35.2147	25.4514	–15.0240	2.6351	1.8567	0.5768
–0.3470	–15.0240	15.1924	–1.9818	–0.8190	–0.3344
–0.1950	2.6351	–1.9818	3.0380	–0.0144	–0.0262
–3.0776	1.8567	–0.8190	–0.0144	0.5151	0.0004
1.9766	0.5768	–0.3344	–0.0262	0.0004	0.8336

P3 =

211.5905	–32.1364	–3.8992	–0.0680	–3.2907	2.8563
–32.1364	24.3479	–16.8871	3.4275	1.6929	0.3585
–3.8992	–16.8871	17.4430	–2.8989	–0.8250	–0.3390
–0.0680	3.4275	–2.8989	4.3828	0.0592	–0.1104
–3.2907	1.6929	–0.8250	0.0592	0.5184	–0.0251
2.8563	0.3585	–0.3390	–0.1104	–0.0251	0.7844

P4 =

208.8743	–35.3788	–0.6676	–0.0678	–3.2899	2.2286
–35.3788	22.9139	–14.9066	2.7662	1.6900	0.5054
–0.6676	–14.9066	15.0331	–2.2374	–0.8202	–0.3361
–0.0678	2.7662	–2.2374	4.3146	–0.0103	–0.0380
–3.2899	1.6900	–0.8202	–0.0103	0.5208	–0.0080
2.2286	0.5054	–0.3361	–0.0380	–0.0080	0.81

Induction motor

P1 =

27.26959318049108	0.49966425825233	–0.05236314612042	0.00015810116375
0.49966425825233	0.02264577349778	–0.00168761549975	0.00000501623009
–0.05236314612042	–0.00168761549975	83.45853436050456	0.49997088632667
0.00015810116375	0.00000501623009	0.49997088632667	0.00809942686912

P2 =

1.0e+002 *

0.27271005742634	0.00499664244014	–0.00062509281117	0.00000171049399
0.00499664244014	0.00020649854859	–0.00001826075651	0.00000005770387
–0.00062509281117	–0.00001826075651	2.87834733289768	0.00499984402924
0.00000171049399	0.00000005770387	0.00499984402924	0.00005653806305

P3 =

27.27546569323679	0.49966440890563	0.08218678077178	0.00025553752961
0.49966440890563	0.02264259313052	–0.00272790341459	0.00001058620793
0.08218678077179	–0.00272790341459	83.47142342809100	0.49997089254736
0.00025553752961	0.00001058620793	0.49997089254736	0.00636228463531

P4 =

1.0e+002 *

0.27274046581253	0.00499664325110	0.00121164697933	0.00000316317406
0.00499664325110	0.00020647900007	–0.00003377301268	0.00000012566275
0.00121164697933	–0.00003377301268	2.87908039728336	0.00499984400694
0.00000316317406	0.00000012566275	0.00499984400694	0.00002428179723