BULGARIAN ACADEMY OF SCIENCES

CYBERNETICS AND INFORMATION TECHNOLOGIES • Volume **10**, No 2

Sofia • 2010

Perturbation Analysis of the LMI-Based Continuous and Discrete-Time Quadratic Stability Problem

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Abstract. *The set of controllers which make closed-loop quadratically stable can be implicitly parametrized by the solutions Q, Y of a system of Linear Matrix Inequalities (LMIs). The paper is concerned with obtaining linear perturbation bounds for the continuous and discrete-time LMI based quadratic stability problem, which are linear functions of the data perturbations. The sensitivity analysis of the perturbed matrix inequalities is considered in a similar manner as for perturbed matrix equations, after introducing a suitable right hand part, which is slightly perturbed. The proposed approach leads to tight linear perturbation bounds for the LMIs' solutions to the quadratic stability problem. Numerical examples are also presented.*

Keywords: Perturbation analysis, quadratic stability problem, LMI based synthesis, Linear systems.

1. Introduction

In many control problems, the design constraints have a simple reformulation in terms of Linear Matrix Inequalities (LMIs). This is hardly surprising, given that LMIs are direct byproducts of Lyapunov based criteria, and that Lyapunov techniques play a central role in the analysis and control of linear systems, see [1, 2]. The analysis of the quadratic stability of a linear system is a good illustration of this point.

The effectiveness of LMI approach remains valuable for several reasons. To begin with it is applicable to all plants without restrictions on infinite or pure imaginary invariant zeros. In addition LMI based design is practical and interesting thanks to the availability of efficient convex optimization algorithms [3] and software [4] plus the MATLAB package Yalmip and SeDuMi solver [5].

In this paper we propose an approach to perform perturbation analysis of the LMI based quadratic stability problem via introducing a suitable right hand part in the considered matrix inequalities. The results obtained after realizing the perturbation analysis can be used in two directions. First it is possible to estimate the errors in the computed solution of the quadratic stability problem, which are due to rounding errors and parametric disturbances in the data. Second it is possible to investigate the robust stability and robust performance of the closed loop system with uncertainty in the plant and in the controller. The uncertainty in the controller appears due to sensitivity of the quadratic stability problem.

We use the following notations: $R^{m\times n}$ – the space of real $m\times n$ matrices; $R^n = R^{n\times 1}$; I_n – the identity $n\times n$ matrix; e_n – the unit $n\times 1$ vector; M^{T} – the transpose of M; M^+ – the pseudo inverse of M; $\|M\|_2=\sigma_{\max}(M)$ – the spectral norm of M, where $\sigma_{\max}(M)$ is the maximum singular value of M; $\mathrm{vec}(M) \in R^{m\times n}$ – the column-wise vector representation of $M \in R^{m\times n}$; $\prod_{m,n} \in R^{mn\times mn}$ – the vec-permutation matrix, such that $\mathrm{vec}(M^{\mathrm{T}}) = \prod_{m,n} \mathrm{vec}(M)$; $M\otimes P$ – the Kroneker product of the matrices M and P. The notation ":=" stands for "equal by definition".

The remainder of the paper is organized as follows. In Section 2 we shortly present the problem set up and objective. Section 3 describes the performed linear sensitivity analysis of the LMI-based continuous and discrete-time quadratic stability problem. Section 4 presents some numerical examples before we conclude in Section 5 with some final remarks.

2. Problem setup and objective

Consider the linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^r$, are the system state, input and output vectors respectively, and A, B are constant matrices of compatible size. It is necessary to find a state-feedback matrix K such that system (1) should be stable. The control input is chosen as $u=Kx$ then the closed loop system is obtained

$$\dot{x}(t) = (A + BK)x(t).$$

To make the Linear Time-Invariant (LTI) system stable it is necessary to use a quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}}Px,\ P > 0,\ P= P^{\mathrm{T}},$$

such that

$$\frac{d}{dt}V(x) = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = x^{\mathrm{T}}[(A+BK)^{\mathrm{T}}P + P(A+BK)]x < 0.$$

In order to ensure quadratic stability of the system (1) the following system of inequalities has to be solved

$$(A + BK)^{\mathrm{T}}P + P(A + BK) < 0,\ P > 0.$$

The system of inequalities is nonlinear with respect to the unknowns P and K that is why we perform linearizing change of variables

$$Q = P^{-1} \Rightarrow P = Q^{-1},\ Y = KP^{-1} = KQ \Rightarrow K = YQ^{-1},$$

to obtain

$$A^{\mathrm{T}}P + K^{\mathrm{T}}B^{\mathrm{T}}P + PA + PBK < 0,$$

$$A^{\mathrm{T}}Q^{-1} + Q^{-1}Y^{\mathrm{T}}B^{\mathrm{T}}Q^{-1} + Q^{-1}A + Q^{-1}BYQ^{-1} < 0.$$

Finally we multiply on left and right the last inequality with Q to obtain a system of LMIs with respect to Q and Y

(2) $$AQ + QA^{\mathrm{T}} + Y^{\mathrm{T}}B^{\mathrm{T}} + BY < 0,\ Q > 0.$$

Then we consider the linear discrete-time system

(3) $$x_{k+1} = Ax_k + Bu_k,$$

Where $x_k \in R^n$, $u_k \in R^m$, and $y_k \in R^r$ are the system state, input and output vectors respectively, and A, B are constant matrices of compatible size. It is necessary to find a state-feedback matrix K such that system (3) should be stable. The control input is chosen as $u_k=Kx_k$ then the closed loop system is obtained

$$x_{k+1} = (A+ BK)x_k.$$

To make the LTI system stable it is necessary to use a quadratic Lyapunov function

$$V(x_{k+1}) = x_k^{\mathrm{T}}Px_k,\ P > 0,\ P = P^{\mathrm{T}},$$

such that

$$V(x_{k+1}) - V(x_k) = x_{k+1}^{\mathrm{T}}Px_k + x_k^{\mathrm{T}}Px_k = x_k^{\mathrm{T}}[(A+ BK)^{\mathrm{T}}P(A+ BK) - P]x_k < 0.$$

In order to ensure quadratic stability of the system (3) the following system of inequalities has to be solved

$$(A + BK)^{\mathrm{T}}P(A+ BK) - P < 0,\ P > 0.$$

The system of inequalities is nonlinear with respect to the unknowns P and K that is why we perform linearizing change of variables

$$Q = P^{-1} \Rightarrow P = Q^{-1},\ Y = KP^{-1} = KQ \Rightarrow K = YQ^{-1},$$

to obtain

$$(A + BK)^{\mathrm{T}}Q^{-1}(A+ BK) - Q^{-1} < 0.$$

Then we multiply on left and right with Q to obtain a system of inequalities, which are not LMIs:

$$(AQ + BY)^{\mathrm{T}} Q^{-1}(AQ + BY) - Q < 0,\ Q > 0.$$

Finally we use Schur complement argument [6] to obtain a system of LMIs with respect to the variables Q and Y

$$\begin{bmatrix} -Q & (AQ+BY)^{\mathrm{T}} \\ AQ+BY & -Q \end{bmatrix} < 0, \quad Q > 0. \tag{4}$$

The main objective of the paper is to perform a linear sensitivity analysis of the LMI systems (2) and (13), needed to solve the continuous and discrete-time quadratic stability problems.

Suppose that the matrices A, B are subject to perturbations ΔA, ΔB and assume that they do not change the sign of the LMI systems (2) and (4). The sensitivity analysis of the continuous and discrete-time LMI based quadratic stability problem is aimed at determining perturbation bounds of the LMIs (2) and (4) as functions of the perturbations in the data A, B.

3. Linear sensitivity analysis

First we perform sensitivity analysis of the LMI (2) for the continuous-time system (1):

$$\begin{gathered}(A + \Delta A)(Q + \Delta Q) + (Q + \Delta Q)(A + \Delta A)^{\mathrm{T}} + \\ + (B + \Delta B)(Y + \Delta Y) + (Y + \Delta Y)^{\mathrm{T}}(B + \Delta B)^{\mathrm{T}} < 0,\end{gathered} \tag{5}$$

and to study the effect of the perturbations $\Delta A, \Delta B$ and ΔY on the perturbed LMI solution $Q^{*}+\Delta Q$, where Q^{*} and ΔQ are the nominal solution of LMI (2) and the perturbation. The essence of our approach is to perform sensitivity analysis of the LMI (2) in a similar manner as for a proper matrix equation after introducing a suitable right hand part, which is slightly perturbed. Thus for the expression (5) we have

$$\begin{gathered}(A+\Delta A)(Q^{*}+\Delta Q)+(Q^{*}+\Delta Q)(A+\Delta A)^{\mathrm{T}} + \\ +(B+\Delta B)(Y^{*}+\Delta Y)+(Y^{*}+\Delta Y)^{\mathrm{T}}(B+\Delta B)^{\mathrm{T}} = M^{*}+\Delta M_1 < 0,\end{gathered} \tag{6}$$

where M^{*} is obtained using the nominal LMI

$$AQ^{*} + Q^{*}A^{\mathrm{T}} + BY^{*} + Y^{*\mathrm{T}}B^{\mathrm{T}} < 0 = M^{*} < 0. \tag{7}$$

The matrix ΔM_1 is due to the data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the LMIs.

Using the relation (7) the perturbed equation (6) may be written as

$$\Delta_Q + \Omega_Q = \Delta M_1, \tag{8}$$

where

$\Delta_Q = A\,\Delta Q + \Delta Q A^{\mathrm{T}}$ and $\Omega_Q = \Delta A Q^* + Q^* \Delta A^{\mathrm{T}} + B\,\Delta Y + \Delta Y^{\mathrm{T}} B^{\mathrm{T}} + \Delta B Y^* + Y^{*\mathrm{T}}\,\Delta B^{\mathrm{T}}$.

Here the terms of second and higher order are neglected. The relation (8) may be written in a vector form as

$$\mathrm{vec}(\Delta_Q) + \mathrm{vec}(\Omega_Q) = \mathrm{vec}(\Delta M_1),$$

where

$$\mathrm{vec}(\Delta_Q) = [I \otimes A + A \otimes I]\,\mathrm{vec}(\Delta_Q):= T\Delta q,$$

here the Lyapunov operator T is invertible for stable matrix A,

$$\mathrm{vec}(\Omega_Q) = [(Q^* \otimes I) + (I \otimes Q^*)\Pi_{n^2},\ (I \otimes B) + (B \otimes I)\Pi_{n\times m},$$

$$(Y^* \otimes I) + (I \otimes Y^{*\mathrm{T}})\Pi_{m^2}] \begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta Y) \\ \mathrm{vec}(\Delta B) \end{bmatrix} = [T_{t1}, T_{t2}, T_{t3}]\Delta_{AYB} := T_t \Delta_{AYB}.$$

Further we obtain the expression

$$T\Delta q + T_{t1}\,\mathrm{vec}(\Delta A) + T_{t2}\,\mathrm{vec}(\Delta Y) + T_{t3}\,\mathrm{vec}(\Delta B) = \mathrm{vec}(\Delta M_1).$$

Finally the relative perturbation bound for the solution Q^* of the LMI (2) has the form

(9)
$$\frac{\|\Delta q\|}{\|\mathrm{vec}(Q^*)\|_2} \le \frac{1}{\|\mathrm{vec}(Q^*)\|_2}\left(T_{AYB1}\frac{\|\mathrm{vec}(\Delta A)\|_2}{\|\mathrm{vec}(A)\|_2} + T_{AYB2}\frac{\|\mathrm{vec}(\Delta Y)\|_2}{\|\mathrm{vec}(Y^*)\|_2}\right) +$$
$$+ \frac{1}{\|\mathrm{vec}(Q^*)\|_2}\left(T_{AYB3}\frac{\|\mathrm{vec}(\Delta B)\|_2}{\|\mathrm{vec}(B)\|_2} + M_1\frac{\|\mathrm{vec}(\Delta M_1)\|_2}{\|\mathrm{vec}(M^*)\|_2}\right),$$

where

$$\frac{T_{AYB1}}{\|\mathrm{vec}(Q^*)\|_2} = \frac{\|T^{-1}\|_2 \|T_{t1}\|_2 \|\mathrm{vec}(A)\|_2}{\|\mathrm{vec}(Q^*)\|_2},\quad \frac{T_{AYB2}}{\|\mathrm{vec}(Q^*)\|_2} = \frac{\|T^{-1}\|_2 \|T_{t2}\|_2 \|\mathrm{vec}(Y^*)\|_2}{\|\mathrm{vec}(Q^*)\|_2},$$

$$\frac{T_{AYB3}}{\|\mathrm{vec}(Q^*)\|_2} = \frac{\|T^{-1}\|_2 \|T_{t3}\|_2 \|\mathrm{vec}(B)\|_2}{\|\mathrm{vec}(Q^*)\|_2},\quad \frac{M_1}{\|\mathrm{vec}(Q^*)\|_2} = \frac{\|T^{-1}\|_2 \|\mathrm{vec}(M^*)\|_2}{\|\mathrm{vec}(Q^*)\|_2}$$

may be considered as individual relative condition numbers of the LMI (2) with respect to the perturbations ΔA, ΔB and ΔY.

In a similar way the relative perturbation bounds for the solution Y^* of the LMI (2) may be obtained using the following expression

(10)
$$\Delta_Y + \Omega_Y = \Delta M_2,$$

where

$$\Delta_Y = B\Delta Y + \Delta Y^{\mathrm{T}} B^{\mathrm{T}}, \text{ and}$$

$$\Omega_Y = A\Delta Q + \Delta Q A^{\mathrm{T}} + \Delta A Q^* + Q^* \Delta A^{\mathrm{T}} + \Delta B Y^* + Y^{*\mathrm{T}} \Delta B^{\mathrm{T}}.$$

Here the terms of second and higher order are neglected. The relation (10) may be written in a vector form as

$$\mathrm{vec}(\Delta_Y) + \mathrm{vec}(\Omega_Y) = \mathrm{vec}(\Delta M_2),$$

where

$$\mathrm{vec}(\Delta_Y) = [I \otimes B + (B \otimes I)\Pi_{n\times m}]\mathrm{vec}(\Delta Y) := W\Delta y,$$

$$\mathrm{vec}(\Omega_Y) = [(Q^* \otimes I) + (I \otimes Q^*)\Pi_{n^2},$$

$$(I \otimes A) + (A \otimes I), (Y^{*\mathrm{T}} \otimes I) + \ldots] \begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta Q) \\ \mathrm{vec}(\Delta B) \end{bmatrix} = [W_{t1}, W_{t2}, W_{t3}]\Delta_{AQB} := W_t \Delta_{AQB}.$$

Further we obtain the expression

$$W\Delta y + W_{t1}\mathrm{vec}(\Delta A) + W_{t2}\mathrm{vec}(\Delta Q) + W_{t3}\mathrm{vec}(\Delta B) = \mathrm{vec}(\Delta M_2).$$

Finally the relative perturbation bound for the solution Y^* of the LMI (2) has the form

$$\begin{aligned} \frac{\|\Delta y\|_2}{\|\mathrm{vec}(Y^*)\|_2} &\le \frac{1}{\|\mathrm{vec}(Y^*)\|_2}\left(W_{AQB1}\frac{\|\mathrm{vec}(\Delta A)\|_2}{\|\mathrm{vec}(A)\|_2} + W_{AQB2}\frac{\|\mathrm{vec}(\Delta Q)\|_2}{\|\mathrm{vec}(Q^*)\|_2}\right) + \\ &+ \frac{1}{\|\mathrm{vec}(Y^*)\|_2}\left(W_{AQB3}\frac{\|\mathrm{vec}(\Delta B)\|_2}{\|\mathrm{vec}(B)\|_2} + M_2\frac{\|\mathrm{vec}(\Delta M_2)\|_2}{\|\mathrm{vec}(M^*)\|_2}\right) \end{aligned} \tag{11}$$

where

$$\frac{W_{AQB1}}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|W^+\|_2\|W_{t1}\|_2\|\mathrm{vec}(A)\|_2}{\|\mathrm{vec}(Y^*)\|_2}, \quad \frac{W_{AQB2}}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|W^+\|_2\|W_{t2}\|_2\|\mathrm{vec}(Q^*)\|_2}{\|\mathrm{vec}(Y^*)\|_2},$$

$$\frac{W_{AQB3}}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|W^+\|_2\|W_{t3}\|_2\|\mathrm{vec}(B)\|_2}{\|\mathrm{vec}(Y^*)\|_2}, \quad \frac{M_2}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|W^+\|_2\|\mathrm{vec}(M^*)\|_2}{\|\mathrm{vec}(Y^*)\|_2}$$

may be considered as individual relative condition numbers of the LMI (2) with respect to the perturbations ΔA, ΔB and ΔQ.

Then we perform sensitivity analysis of the LMI (4) for the discrete-time system (3):

$$\begin{bmatrix} -(Q+\Delta Q) & (Q+\Delta Q)(A+\Delta A)^{\mathrm{T}} + (Y+\Delta Y)^{\mathrm{T}}(B+\Delta B)^{\mathrm{T}} \\ (A+\Delta A)(Q+\Delta Q) + (B+\Delta B)(Y+\Delta Y) & -(Q+\Delta Q) \end{bmatrix} < 0, \tag{12}$$

and to study the effect of the perturbations ΔA, ΔB and ΔY on the perturbed LMI solution $Q^* + \Delta Q$, where Q^* and ΔQ are the nominal solution of LMI (4) and the perturbation. The essence of our approach is to perform sensitivity analysis of the LMI (4) in a similar manner as for a proper matrix equation after introducing a suitable right hand part, which is slightly perturbed. Thus for the expression (12) we have

$$\begin{aligned} &\begin{bmatrix} -(Q^*+\Delta Q) & (Q^*+\Delta Q)(A+\Delta A)^{\mathrm{T}} + (Y+\Delta Y)^{\mathrm{T}}(B+\Delta B)^{\mathrm{T}} \\ (A+\Delta A)(Q^*+\Delta Q) + (B+\Delta B)(Y^*+\Delta Y) & -(Q^*+\Delta Q) \end{bmatrix} = \\ &= N^* + \Delta N_1 < 0, \end{aligned} \tag{13}$$

where N^* is obtained using the nominal LMI

(14) $$\begin{bmatrix} -Q^* & Q^*A^{\mathrm{T}} + Y^{*\mathrm{T}}B^{\mathrm{T}} \\ AQ^* + BY^* & -Q^* \end{bmatrix} = N^* < 0.$$

The matrix ΔN_1 is due to the data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the LMIs.

Using the relation (14) the perturbed equation (13) may be written as

(15) $$\Delta_Q + \Omega_Q = \Delta N_1,$$

where

$$\Delta_Q := \begin{bmatrix} -\Delta Q & \Delta Q A^{\mathrm{T}} \\ A\Delta Q & -\Delta Q \end{bmatrix},$$

$$\Omega_Q := \begin{bmatrix} 0 & Q^*\Delta A^{\mathrm{T}} + \Delta Y^{\mathrm{T}}B^{\mathrm{T}} + Y^{*\mathrm{T}}\Delta B^{\mathrm{T}} \\ \Delta A Q^* + B\Delta Y + \Delta B Y^* & 0 \end{bmatrix}.$$

Here the terms of second and higher order are neglected. The relation (15) may be written in a vector form as

$$\mathrm{vec}(\Delta_Q) + \mathrm{vec}(\Omega_Q) = \mathrm{vec}(\Delta N_1),$$

where

$$\mathrm{vec}(\Delta_Q) = \begin{bmatrix} -I \\ A\otimes I \\ I\otimes A \\ -I \end{bmatrix} \mathrm{vec}(\Delta Q) := V\Delta q,$$

$$\mathrm{vec}(\Omega_Q) = \begin{bmatrix} 0 & 0 & 0 \\ (I\otimes Q^*)\Pi_{n^2} & (B\otimes I)\Pi_{n\times m} & (I\otimes Y^{*\mathrm{T}})\Pi_{m^2} \\ Q^*\otimes I & I\otimes B & Y^*\otimes I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta Y) \\ \mathrm{vec}(\Delta B) \end{bmatrix} =$$

$$= [V_{t1}, V_{t2}, V_{t3}]\Delta_{AYB} := V_t\Delta_{AYB}.$$

Further we obtain the expression

$$V\Delta q + V_{t1}\mathrm{vec}(\Delta A) + V_{t2}\mathrm{vec}(\Delta Y) + V_{t3}\mathrm{vec}(\Delta B) = \mathrm{vec}(\Delta N_1).$$

Finally the relative perturbation bound for the solution Q^* of the LMI (4) has the form

(16) $$\frac{\|\Delta q\|_2}{\|\mathrm{vec}(Q^*)\|_2} \le \frac{1}{\|\mathrm{vec}(Q^*)\|_2}\left(V_{AYB1}\frac{\|\mathrm{vec}(\Delta A)\|_2}{\|\mathrm{vec}(A)\|_2} + V_{AYB2}\frac{\|\mathrm{vec}(\Delta Y)\|_2}{\|\mathrm{vec}(Y^*)\|_2} \right) + \\ + \frac{1}{\|\mathrm{vec}(Q^*)\|_2}\left(V_{AYB3}\frac{\|\mathrm{vec}(\Delta B)\|_2}{\|\mathrm{vec}(B)\|_2} + N_1\frac{\|\mathrm{vec}(\Delta N_1)\|_2}{\|\mathrm{vec}(N^*)\|_2} \right)$$

where

$$\frac{V_{AYB1}}{\| \mathrm{vec}(Q^*) \|_2} := \frac{\| V^+ \|_2 \| V_{t1} \|_2 \| \mathrm{vec}(A) \|_2}{\| \mathrm{vec}(Q^*) \|_2}, \ \frac{V_{AYB2}}{\| \mathrm{vec}(Q^*) \|_2} := \frac{\| V^+ \|_2 \| V_{t2} \|_2 \| \mathrm{vec}(Y^*) \|_2}{\| \mathrm{vec}(Q^*) \|_2},$$

$$\frac{V_{AYB3}}{\| \mathrm{vec}(Q^*) \|_2} := \frac{\| V^+ \|_2 \| V_{t3} \|_2 \| \mathrm{vec}(B) \|_2}{\| \mathrm{vec}(Q^*) \|_2}, \ \frac{N_1}{\| \mathrm{vec}(Q^*) \|_2} := \frac{\| V^+ \|_2 \| \mathrm{vec}(N^*) \|_2}{\| \mathrm{vec}(Q^*) \|_2}$$

may be considered as individual relative condition numbers of the LMI (4) with respect to the perturbations ΔA, ΔB, and ΔY.

In a similar way the relative perturbation bounds for the solution Y^* of the LMI (4) may be obtained using the following expression

$$\Delta_Y + \Omega_Y = \Delta N_2, \tag{17}$$

where

$$\Delta_Y := \begin{bmatrix} 0 & \Delta Y^{\mathrm{T}} B^{\mathrm{T}} \\ B\Delta Y & 0 \end{bmatrix},$$

$$\Omega_Y := \begin{bmatrix} -\Delta Q & \Delta Q A^{\mathrm{T}} + Q^* \Delta A^{\mathrm{T}} + Y^{*\mathrm{T}} \Delta B^{\mathrm{T}} \\ A\Delta Q + \Delta A Q^* + \Delta B Y^* & -\Delta Q \end{bmatrix}.$$

Here the terms of second and higher order are neglected. The relation (17) may be written in a vector form as

$$\mathrm{vec}(\Delta_Y) + \mathrm{vec}(\Omega_Y) = \mathrm{vec}(\Delta N_2),$$

where

$$\mathrm{vec}(\Delta_Y) = \begin{bmatrix} 0 \\ (B \otimes I)\Pi_{n\times m} \\ I \otimes B \\ 0 \end{bmatrix} \mathrm{vec}(\Delta Y) := U \Delta y,$$

$$\mathrm{vec}(\Omega_Y) = \begin{bmatrix} 0 & -I & 0 \\ (I \otimes Q^*)\Pi_{n^2} & A \otimes I & (I \otimes Y^{*\mathrm{T}})\Pi_{m^2} \\ Q^* \otimes I & I \otimes A & Y^* \otimes I \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta Q) \\ \mathrm{vec}(\Delta B) \end{bmatrix} =$$

$$= [U_{t1}, U_{t2}, U_{t3}] \Delta_{AQB} := U_t \Delta_{AQB}.$$

Further we obtain the expression

$$U\Delta y + U_{t1}\mathrm{vec}(\Delta A) + U_{t2}\mathrm{vec}(\Delta Q) + U_{t3}\mathrm{vec}(\Delta B) = \mathrm{vec}(\Delta N_2).$$

Finally the relative perturbation bound for the solution Y^* of the LMI (4) has the form

$$\text{(18)}\quad \frac{\|\Delta y\|_2}{\|\mathrm{vec}(Y^*)\|_2} \le \frac{1}{\|\mathrm{vec}(Y^*)\|_2}\left(U_{AQB1}\frac{\|\mathrm{vec}(\Delta A)\|_2}{\|\mathrm{vec}(A)\|_2} + U_{AQB2}\frac{\|\mathrm{vec}(\Delta Q)\|_2}{\|\mathrm{vec}(Q^*)\|_2}\right) + \\ + \left(U_{AYB3}\frac{\|\mathrm{vec}(\Delta B)\|_2}{\|\mathrm{vec}(B)\|_2} + N_2\frac{\|\mathrm{vec}(\Delta N_2)\|_2}{\|\mathrm{vec}(N^*)\|_2}\right)$$

where

$$\frac{U_{AYB1}}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|U^+\|_2\|U_{t1}\|_2\|\mathrm{vec}(A)\|_2}{\|\mathrm{vec}(Y^*)\|_2},\quad \frac{U_{AYB2}}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|U^+\|_2\|U_{t2}\|_2\|\mathrm{vec}(Q^*)\|_2}{\|\mathrm{vec}(Y^*)\|_2},$$

$$\frac{U_{AYB3}}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|U^+\|_2\|U_{t3}\|_2\|\mathrm{vec}(B)\|_2}{\|\mathrm{vec}(Y^*)\|_2},\quad \frac{N_2}{\|\mathrm{vec}(Y^*)\|_2} = \frac{\|U^+\|_2\|\mathrm{vec}(N^*)\|_2}{\|\mathrm{vec}(Y^*)\|_2}$$

may be considered as individual relative condition numbers of the LMI (4) with respect to the perturbations ΔA, ΔB, and ΔQ.

4. Numerical examples

Consider the continuous-time system (1), where

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix},\quad B = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -pm & -pc/m & -pk/m & 1/m \end{vmatrix},$$

$$C = \begin{bmatrix} -k/m & -c/m \\ 0 & c \\ k & 0 \\ 1 & 0 \end{bmatrix},\quad D = \begin{bmatrix} -pm & -pc/m & -pk/m & 1/m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $m = 3$, $c = 1$, $k = 2$, $pm = 0.4$, $pc = 0.2$, $pk = 0.3$.

The perturbations in the system matrices of the continuous-time system are chosen as

$$\Delta A = A \times 10^{-i},\ \Delta B = B \times 10^{-i},\ \Delta C = C \times 10^{-i},\ \Delta D = D \times 10^{-i},$$

$$\Delta M_1 = M^* \times 10^{-i},\ \Delta M_2 = M^* \times 10^{-i},\ \Delta Q^* = Q^* \times 10^{-i},\ \Delta Y = Y^* \times 10^{-i}.$$

The perturbed solutions $Q^* + \Delta Q$ and $Y^* + \Delta Y$ are computed based on the method derived in [7] and using the software [4]. The relative perturbation bounds for the solutions Q^* and Y^* of the LMIs (2) are obtained by the linear bounds (9) and (11), respectively.

The results obtained for different values of i are shown in the Table 1.

Table 1

i	$\frac{\lVert \Delta q \rVert_2}{\lVert \mathrm{vec}(Q^*) \rVert_2}$	Bound (9)	$\frac{\lVert \Delta y \rVert_2}{\lVert \mathrm{vec}(Y^*) \rVert_2}$	Bound (11)
8	7.9616×10^{-8}	3.0119×10^{-7}	2.9520×10^{-8}	5.2288×10^{-8}
7	7.9616×10^{-7}	3.0119×10^{-6}	2.9520×10^{-7}	5.2288×10^{-7}
6	7.9616×10^{-6}	3.0119×10^{-5}	2.9520×10^{-6}	5.2288×10^{-6}
5	7.9616×10^{-5}	3.0119×10^{-4}	2.9520×10^{-5}	5.2288×10^{-5}
4	7.9616×10^{-4}	3.0119×10^{-3}	2.9520×10^{-4}	5.2288×10^{-4}

The obtained perturbation bounds (9) and (11), based on the presented solution approach, are close to the real relative perturbation bounds $\frac{\lVert \Delta q \rVert_2}{\lVert \mathrm{vec}(Q^*) \rVert_2}$ and $\frac{\lVert \Delta y \rVert_2}{\lVert \mathrm{vec}(Y^*) \rVert_2}$, thus they are good in sense that they are tight.

Consider the discrete-time system (3), where

$$A = \begin{bmatrix} 1 & 0.01 \\ -0.0067 & 0.9966 \end{bmatrix}, \quad B = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -0.004 & -0.0007 & -0.001 & 0.0033 \end{vmatrix},$$

$$C = \begin{bmatrix} -0.6667 & -0.333 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -0.4 & -0.0667 & -0.1 & 0.333 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The perturbations in the system matrices of the discrete-time system are chosen as

$$\Delta A = A \times 10^{-i},\ \Delta B = B \times 10^{-i},\ \Delta C = C \times 10^{-i},\ \Delta D = D \times 10^{-I},$$
$$\Delta Q^* = Q^* \times 10^{-i},\ \Delta Y = Y^* \times 10^{-i}.$$

The perturbed solutions $Q^* + \Delta Q$ and $Y^* + \Delta Y$ are computed based on the method derived in [7] and using the software [4]. The relative perturbation bounds for the solutions Q^* and Y^* of the LMIs (4) are obtained by the linear bounds (16) and (18), respectively.

The results obtained for different values of i are shown in the Table 2.

Table 2

i	$\frac{\lVert \Delta q \rVert_2}{\lVert \mathrm{vec}(Q^*) \rVert_2}$	Bound (16)	$\frac{\lVert \Delta y \rVert_2}{\lVert \mathrm{vec}(Y^*) \rVert_2}$	Bound (18)
8	1.6960×10^{-8}	3.5967×10^{-8}	1.0338×10^{-8}	5.0296×10^{-8}
7	1.6960×10^{-7}	3.5967×10^{-7}	1.0338×10^{-7}	5.0296×10^{-7}
6	1.6960×10^{-6}	3.5967×10^{-6}	1.0338×10^{-6}	5.0296×10^{-6}
5	1.6960×10^{-5}	3.5967×10^{-5}	1.0338×10^{-5}	5.0296×10^{-5}
4	1.6960×10^{-4}	3.5967×10^{-4}	1.0338×10^{-4}	5.0296×10^{-4}

The obtained perturbation bounds (16) and (18), based on the presented solution approach, are close to the real relative perturbation bounds $\frac{\| \Delta q \|_2}{\| \mathrm{vec}(Q^*) \|_2}$ and $\frac{\| \Delta y \|_2}{\| \mathrm{vec}(Y^*) \|_2}$, thus they are good in sense that they are tight.

5. Conclusion

The linear sensitivity analysis of the continuous and discrete-time LMI based quadratic stability problem has been studied. Tight perturbation bounds, which are linear functions of the data perturbations, have been obtained for the matrix inequalities determining the problem solution. Based on these results we have presented numerical examples to explicitly reveal the performance and applicability of the proposed approach to analyze the sensitivity of the LMI based quadratic stability problem.

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