Multiobjective Optimization Methods Help to Minimize a Function over the Efficient Set

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Abstract. *The problem for obtaining an upper bound for the minimal value of a linear function on the efficient set of a MOLP problem is considered. A multiobjective optimization method is used to get Pareto (or efficient) points. An illustrative example is presented.*

Keywords. *Multiple objective optimization, efficient set, ε-constraint method.*

1. Introduction

The multiobjective linear programming (MOLP) problem can be described as follows:

$$\max f_1(x)$$
$$\ldots$$
(1)
$$\ldots$$
$$\ldots$$
$$\max f_m(x)$$

s.t.

$$x \in S \subseteq R^n$$

Here $f_i(x)$, $i = 1, 2, \ldots, m$, are linear functions. The vector $x \in S$ is called an argument vector. The vector $f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \in R^m$ is called a criteria vector. The feasible set S is defined as follows :

$$S = \{x \in R^n \mid c_i(x) \le 0, \ i = 1, 2, \ldots, k\}.$$

All $c_i(x)$ are linear functions, too. The list of constraints $c_i(x) \le 0$, $i = 1, 2, \ldots, k$, contains the inequalities $x_j \ge 0$ for all $j = 1, 2, \ldots, n$. We suppose that the set S is not empty and is bounded. The set

$$Z = \{z \in R^m \mid z = f(x), \ x \in S\}$$

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is called an attainable set in the criteria space. The point $z^1 = f(x^1) \in Z$, $x^1 \in S$, is called a nondominated (Pareto) point, if there does not exist another point $x^2 \in S$, $x^2 \neq x^1$, such that the following two conditions are fulfilled simultaneously :

$$f_i(x^2) \geq f_i(x^1) \quad \text{for all } i = 1, 2, \ldots, m;$$
$$f_j(x^2) > f_j(x^1) \quad \text{for one } j \text{ at least.}$$

If the point $z^1 = f(x^1)$, $z^1 \in Z$, is nondominated , then the point $x^1 \in S$ is called an efficient point. The set $P \subseteq Z$, containing all nondominated points from Z, is called a nondominated set (Pareto set). The set $E \subseteq S$, containing all efficient points from S, is called an efficient set. For each MOLP problem the set E is closed.

There is also the notion of weak efficiency. The point $x^1 \in S$ is called weakly efficient if there does not exist another point $x^2 \in S$, such that $f_i(x^2) > f_i(x^1)$ for all i. By analogy with the text above we have the notions of weakly efficient set (subset of S) and weakly nondominated (weakly Pareto) set (subset of $f(S)$).

In addition we have given the function $\varphi(x)$, that is a linear function on S. Having in mind problem (1), our purpose in this paper will be to describe a way for obtaining an upper bound for the following minimum :

(2)
$$\min_{x \in E} \varphi(x) = B.$$

This is not a standard mathematical programming problem because the set E is not convex.

2. A short review of the literature

Many papers in various journals consider the problems for optimization of a function over the set E. Some of the first steps in this direction were connected with the idea to organize a movement in the set of efficient extreme points of S only. Later on many attempts have been made to apply various optimization techniques for solving or analyzing problem (2). The survey of Y a m a m o t o [21] proposes a classification of the existing algorithms for optimization over the efficient set. This classification contains seven classes: adjacent vertex search algorithms, nonadjacent vertex search algorithms, face search algorithms, branch and bound search algorithms, Lagrangean relaxation based algorithms, dual approach, bisection algorithms. Yamamoto's paper contains some information about the considered algorithms concerning the corresponding compu-tational requirements.

T h i, P h a m and T h o a i [14] propose a branch and bound procedure based on some properties in Lagrange duality. They give a way to obtain a global search algorithm. The paper contains data about computational experiments on a large set of examples.

Y a m a d a, T a n i n o, I n u i g u c h i [22] propose a method for approximate minimization of a convex function over the weakly efficient set. The method uses a branch and bound procedure.

For a similar problem H o r s t and T h o a i [8] propose again an algorithm of branch and bound type.

A bisecting search algorithm is proposed by H. B e n s o n [2].

A penalty function approach to maximize a function over the efficient set is proposed by D. J. W h i t e [20].

Linear functions optimization on an integer efficient set is considered in the paper of A b b a s and C h a a b a n e [1].

It is worth noting that the relevant literature does not contain many data about applications of multiobjective optimization methods for analysis of problem (2).This paper gives some information of this type.

3. Some preliminary information

In this paper the ε-constraint method is chosen with the purpose to find efficient points in S. The main optimization construction of this method is briefly described here.

We choose in an arbitrary manner the function f_p for maximization and we convert the rest part of functions f_i to inequality constraints. So we obtain the following single objective programming problem

$$(3) \qquad \qquad \max \ f_p(x)$$
$$\text{s.t.} \qquad f_j(x) \geq r_j \quad (\text{for all } j, \text{ but } j \neq p),$$
$$x \in S.$$

Every optimal solution of problem (3) is weakly efficient in the original multi-objective problem (1) (E h r g o t t and G a n d i b l e u x [6] , C h a n k o n g and H a i m e s [4], M i e t t i n e n [12]).

The point $x^* \in S$ is efficient if and only if it is a solution to this problem for every $p = 1, \ldots, m$, and $r_j = f_j(x^*)$ $(j = 1, \ldots, m, j \neq p)$ (M i e t t i n e n [12]). For each given vector $r = (r_1, \ldots, r_{p-1}, r_{p+1}, \ldots, r_m)$ an optimal solution of the ε-constraint problem (3) is efficient if it is unique ((M i e t t i n e n [12], E h r g o t t and G a n d i b l e u x [6], C h a n k o n g and H a i m e s [4]). We will suppose for our purposes that $r \in f(S)$. Many other data about the properties of this problem can be found in the works cited above.

Now we will introduce the notion of *wall* of the set S. In problem (1) the set S is described by the constraints $c_i(x) \leq 0, i = 1, 2, \ldots, k,$ and this list contains the inequalities $x_j \geq 0$ for all $j = 1, 2, \ldots, n$. In addition, this list does not contain redundant constraints. Let's consider the sets W_j where

$$(4) \qquad \qquad W_j = \{x \in S \mid c_j(x) = 0\}, j = 1, 2, \ldots, p \ .$$

Each one of these sets is called a *wall* of the set S . It must be noted that there is a more general notion of a *facet*. A definition of this notion can be found in Steuer [13]. So each wall is a facet, but there can exist a facet that is not a wall.

It is well known in the theory of MOLP problems that if an inner point of S is efficient, then all points of S are efficient (S t e u e r, [13]). So we will suppose that all efficient points of S belong to the frontier of this set, i.e. each efficient point of S belongs to one of its walls, at least.

4. The main idea for obtaining efficient points with "small" value of φ

The main idea of this paper is presented here. The efficient set belongs to the frontier of S. (Correspondingly the Pareto set belongs to the frontier of $f(S)$). Therefore it is sufficient to examine all walls of S (one by one). We need to find points that: belong to some wall, belong to the set E and that have corresponding "small" values of φ. To do this we firstly determine a subset of S, that is described by linear constraints, "close" to the "studied" wall and with the same form; this subset "is shifting" the studied wall a little bit inside. We minimize the function φ on the so obtained subset. As a result we obtain a point $r \in f(S)$ and we use this point in the constraints of ε-constraint problem (3). We solve the so modified problem (3) sequentially for all criteria f_i. If the obtained point belongs to the "studied" wall, it has a corresponding "small" value of φ. The Pareto property is checked for all obtained points of criteria space. If we denote by C the minimal value among all values of φ obtained by this procedure, we have the following inequality

$$\min_{x \in E} \varphi(x) \leq C.$$

5. Some experiments

To illustrate the proposed method of computations we will consider the following MOLP example (S t e u e r [13, Example 8, p. 244])

$$\max f_1(x),$$
$$\max f_2(x),$$
$$\max f_3(x),$$

s.t.

$$x \in S,$$

where:

$$f_1(x) = x_1 + 3x_2 - 2x_3 + x_5;$$
$$f_2(x) = 3x_1 - x_2 + 3x_4 + x_5;$$
$$f_3(x) = x_1 + 2x_3 + 3x_5.$$

The set S is described by the following constraints:

$$2x_1 + 4x_2 + 3x_5 + SL_1 = 27,$$
$$2x_3 + 5x_4 + 4x_5 + SL_2 = 35,$$
$$5x_1 + SL_3 = 26,$$
$$2x_4 + SL_4 = 24,$$
$$5x_1 + 5x_2 + 2x_3 + SL_5 = 36.$$

Here SL_i are the slack variables. The following linear function is to be minimized

$$\varphi(x) = 3x_1 + 5x_2 - 4x_3 - 2x_4 + 3x_5.$$

The data given by Steuer contain the list of the nondominated extreme points, belonging to the criteria space $f(S)$. The recalculated coordinates of these points are presented in Table 1.

Table 1

A nondominated extreme point	f_1	f_2	f_3	φ
z^1	20.25	14.25	0.0	19.75
z^2	19.8	17.4	0.9	20.2
z^3	9.3125	8.5625	26.25	27.1875
z^4	14.0666	30.5866	13.80	24.7865
z^5	9.125	9.875	26.625	27.375
z^6	10.73333	28.85333	21.80	27.05332
z^7	11.2	34.6	5.2	11.6
z^8	−1.26083	20.26083	34.04343	8.086998
z^9	5.2	36.6	5.2	1.6
z^{10}	0.733333	22.853333	31.80	11.05333
z^{11}	−34.80	0.6	35.2	−69.4

The last column of Table 1 contains the corresponding values of function φ. It can be seen that the proposed method of computations has to find the value −69.4 or a good approximation.

There is an observation that $\max_{x \in S}(2x_4) = 14$, therefore $\min_{x \in S} SL_4 = 10$. At the same time $\min_{x \in S} SL_i = 0$, $i = 1, 2, 3, 5$.

We shall not shift all walls of S. For demonstration purposes we shall consider the following subsets only:

$$Q_1 = \{x \in S \mid 2x_1 + 4x_2 + 3x_5 = 26.9\},$$
$$Q_2 = \{x \in S \mid 2x_3 + 5x_4 + 4x_5 = 34.9\},$$
$$Q_3 = \{x \in S \mid 5x_1 = 25.9\},$$
$$Q_4 = \{x \in S \mid 2x_4 = 14\},$$
$$Q_5 = \{x \in S \mid 5x_1 + 5x2 + 2x_3 = 35.9\}.$$

Minimizing the function φ on each one of these subsets we obtain the results presented in Table 2.

Table 2

Subset	Obtained minimal value of φ	Corresponding vector of $f(S)$
$x \in Q_1$	−5.854839	$r^1 = (−3.248387, 0.641935, 30.79032)$
$x \in Q_2$	−69.8	$r^2 = (−34.9, 0.0, 34.9)$
$x \in Q_3$	−14.62	$r^3 = (−4.92, 30.48, 15.28)$
$x \in Q_4$	−14.0	$r^4 = (0.0, 21, 0.0)$
$x \in Q_5$	−69.46	$r^5 = (−34.82, 0.54, 35.18)$

Solving the corresponding ε-constraint problem (3) for each r^i we obtain the results presented in Table 3.

All vectors written in the third column of this table are Pareto vectors. It is clear that in the considered case the proposed method gives the needed value **exactly**. The corresponding vector $x^* \in E$ is

$$x^* = (0.2, 0.0, 17.5, 0.0, 0.0).$$

Table 3

Used vector r^i	Function f_i that is maximized	Obtained Pareto vector	Corresponding value of φ
r^1	f_1	3.293552, 15.80645, 30.79032	16.54517
	f_2	1.743013, 23.45914, 30.79032	12.66882
	f_3	−3.248387, 19.09577, 34.11201	3.495106
r^2	f_1	−26.1, 5.7, 34.9	−49.3
	f_2	−26.1, 5.7, 34.9	−49.3
	f_3	−34.8, 0.6, 35.2	**−69.4**
r^3	f_1	12.8, 30.48, 15.28	24.136
	f_2	8.56, 31.89, 15.28	17.056
	f_3	9.571429, 30.48, 18.31428	21.70857
r^4	f_1	18.23478, 21, 4.421739	21.45217
	f_2	5.2, 36.6, 5.2	1.6
	f_3	0.0, 21, 32.78571	10.10714
r^5	f_1	−34.22, 0.94, 35.18	−68.06
	f_2	−34.22, 0.94, 35.18	−68.06
	f_3	−34.8, 0.6, 35.2	**−69.4**

But generally it is of interest to know whether there is another efficient point x_a, such that $\varphi(x_a) \leq -69.6$. If not, we have a lower bound for min $\varphi(x)$, $x \in E$.

The set

$$\{x \in S \,|\, \varphi(x) \leq -69.6\}$$

has the following ideal point: (−34.75; 0.4; 35.1333) and this is an attainable point in $f(S)$. We can use this point as a reference point and using the reference point method of Prof. Wierzbicki [18, 19] we obtain that it is dominated by the point (−34.68719; 0.66613; 35.19611), that is an attainable and Pareto point. This means that the checked set $\{x \in S \,|\, \varphi(x) \leq -69.6\}$ does not contain any efficient points. Thus we have the following two inequalities:

$$-69.6 \leq \min_{x \in E} \varphi(x) \leq -69.4.$$

They can be obtained without any preliminary data about the efficient set E of the considered example.

A LINGO-9 DEMO version is used for all computations here presented.

6. Conclusion

Using the described procedure for obtaining an upper bound for B we have obtained the exact value practically in all experiments. This procedure needs standard LP software only. This procedure does not need as initial data any results obtained by some earlier developed methods for optimization over the efficient set. It does not use any methods for efficient set investigation. It does not use any special optimization methods. The properties of the approach here proposed allow the use of parallel computing. Now there is a need of a good corresponding method for obtaining lower bounds of B. This will confirm the opinion that the multiobjective optimization techniques can have some influence on the methods for optimization over the efficient set.

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