SPATIAL FACE RACK DRIVES: MATHEMATICAL MODELS FOR SYNTHESIS AND SOFTWARE ILLUSTRATIONS

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ABSTRACT
Spatial three-link face rack mechanisms are applied to realize rotation transformation of one of the movable links (pinion) into rectilinear transformation of the other link (gear rack). The principle for the synthesis upon mesh region is applied in this study by developing an adequate mathematical model. Here are shown algorithms, analytically describing active tooth surfaces of rotating link, mesh region and the link which realize translation motion. The results, which treat synthesis of spatial convolute, Archimedean and involute face rack drives, are illustrated graphically.

Keywords: mathematical modeling, synthesis, face linear helicoid, face rack drive.

INTRODUCTION
Spatial three links transmissions, transforming motions through a system of high kinematic joints, in accordance with preliminary given law (velocity ratio) and randomly oriented in the space vectors of motion of the movable links, are characterized by the biggest variety and complexity of their study. Therefore they are still insufficiently studied and thus their exploitation and technological capabilities are not realized completely (Abadjiev, 2007), (Abadjieva, 2010). This work contains special type mechanisms, belonging to transformers of rotation into translation motions (vice versa), by a system of high kinematic joints. These mechanical transmissions are known as a rack drives (Gavrilenko, 1971).

In many well-known literature sources (Litvin, 1994), rack matings are treated as instrumental matings, when cylindrical involute gears are cut. Planar mechanical motion transformers – rack cylindrical gear drives are subject of study. The article (Tarhanov, 1973) is dedicated to a special rack drive, consisting of a bevel gear and a rack with straight teeth. This mechanism realizes a variable law of motion transformation. In publication (Watanabe, 2006) analysis and practical experiment of rack set of type Wildhaber-Novikov (WN-helical gear rack drive) is accomplished. The monograph (Abadjieva, 2011) should be considered as a development and improvement of the studies performed in (Abadjiev, 2007), (Abadjieva, 2010), (Kovatchev, 1990), (Abadjieva, 2009).

This study presents a mathematical model for synthesis upon mesh region of a face rack set, in which the active tooth surfaces of the rotating link - face pinion are parts of face linear helicoids and tooth surfaces of the link, performing rectilinear translation- face rack are theoretically conjugated with those of the face pinion. The rotation axis of the face pinion is orthogonal crossed with the direction of the rectilinear translation of the gear rack.
SYNTHESIS OF FACE RACK MECHANISMS

Face convolute helicoids generation. The process of face convolute helicoid $\Sigma_f^{(j)} \quad (j = 1, 2)$ generation is studied in coordinate system $S_f^{(j)}(O_f^{(j)}, x_f^{(j)}, y_f^{(j)}, z_f^{(j)})$(Fig. 1). The generatrix line $L_f^{(j)}$ does not intersect the axis $O_f^{(j)}z_f^{(j)}$ and concludes with it an angle $\pi/2 < \xi^{(j)} < \pi$.

The smallest distance between the $O_f^{(j)}z_f^{(j)}$ and $L_f^{(j)}$ is $r_0^{(j)}$. The $r_0^{(j)}$ is the radius-vector of the directed cylinder, for which the condition $r_0^{(j)} = constant$ is fulfilled for the entire process of face convolute helicoid $\Sigma_f^{(j)}$ generation. The generation of the $\Sigma_f^{(j)}$ is in result of the crossed (tangential related to the directed cylinder $C^{(j)}$) helical motion of the line $L_f^{(j)}$ with a helical parameter $t^{(j)} = constant$. The generatrix $L_f^{(j)}$ lies in a plane $T^{(j)}$, that is tangential to the directed cylinder $C^{(j)}$. Used here and below index $j$ obtains values $j = 1, 2$: $j = 1$ is referred to the parameters, related to the face linear helicoids $\Sigma_f^{(1)}$ generation, $j = 2$ – to the parameters referred to the face helicoid $\Sigma_f^{(2)}$ generation. Parts of surfaces $\Sigma_f^{(1)}$ and $\Sigma_f^{(2)}$ can be used as active tooth surfaces of the face worm (face hob, respectively). The active tooth surfaces $\Sigma_f^{(1)}$ are placed closer to the $O_f^{(j)}z_f^{(j)}$ then the active tooth surfaces $\Sigma_f^{(2)}$ are. In this study, in the process of $\Sigma_f^{(j)} \quad (j = 1, 2)$ generation, for simplicity as well as from technological considerations, related to the real gear cutting, is accepted $C^{(j)} \equiv C$, i.e. $r_0^{(j)} = r_0^{(2)} = r_0, \ T^{(j)} \equiv T, \ \vartheta^{(j)} = \vartheta^{(2)} = \vartheta$ and $t^{(1)} = t^{(2)} = p_t = constant$.

Let point $N^{(j)}$ belongs to the line $L_f^{(j)}$, that generates by its helical motion the helicoid $\Sigma_f^{(j)}$. Then for the vector equation of $\Sigma_f^{(j)}$ is written:

$$\overline{p}_f^{(j)} = \overline{r}_0 + \overline{t} + \overline{u}^{(j)},$$

where

$$t = p_t \vartheta \quad \text{is the value of the crossed (tangential) helical motion of the line } L_f^{(j)};$$

$$u^{(j)}, \ \vartheta \quad \text{are the independent coordinates of } \Sigma_f^{(j)}.$$  

The equality (1) is written in the coordinate system $S_f^{(j)}$ and it is obtained:

$$x_f^{(j)} = r_0 \cos \vartheta - (u^{(j)} \sin \varphi^{(j)} - p_t \vartheta) \sin \vartheta,$$

$$y_f^{(j)} = r_0 \sin \vartheta + (u^{(j)} \sin \varphi^{(j)} - p_t \vartheta) \cos \vartheta,$$

$$z_f^{(j)} = \mp u^{(j)} \cos \varphi^{(j)}.$$ (2)

Substituting in (2)

$$R_0^{(j)} = u^{(j)} - \frac{p_t \vartheta}{\sin \varphi^{(j)}}, \quad p^{(j)} = -p_t \cot \varphi^{(j)} > 0,$$

the equations system (2) can be written as:
The equalities (4) can be treated as a system of equations describing cylindrical helicoids with axial helical parameter $p^{(j)} = \text{constant}$.

**Face involute helicoids generation.** As it has been mentioned the main characteristic of the surfaces, whose generatrix is a straight-line, is their parameter of distribution. For determining the parameter of distribution let present (4) as follow:

$$\begin{align*}
\bar{p}_f^{(j)} &= \bar{p}_0^{(j)} + R_0^{(j)} T^{(j)}. 
\end{align*}$$

(5)

Here $\bar{p}_0^{(j)} = \bar{p}_0^{(j)}(\vartheta)\{\chi_x^{(j)}, \chi_y^{(j)}, \chi_z^{(j)}\}$ can be considered as an equation of the directed helical line, situated on the directed cylinder $C$, and $T^{(j)}\{l_x^{(j)}, l_y^{(j)}, l_z^{(j)}\}$ is a directed vector of the generatrix line $L^{(j)}$. It is obvious, that:

$$\begin{align*}
\chi_x^{(j)} &= r_0 \cos \vartheta, & \chi_y^{(j)} &= r_0 \sin \vartheta, & \chi_z^{(j)} &= \pm p^{(j)} \vartheta, \\
l_x^{(j)} &= -\sin \xi^{(j)} \cos \vartheta, & l_y^{(j)} &= \sin \xi^{(j)} \sin \vartheta, & l_z^{(j)} &= \mp \cos \xi^{(j)}. 
\end{align*}$$

(6)

It is known (Rashevsky, 1956), that the parameter of distribution of the face linear helicoids is determined, by using the following relation:
\[ h^{(j)} = \frac{[d\vec{P}_0^{(j)}]^{(j)} d\vec{f}^{(j)}}{(d\vec{l}^{(j)})^2} = \begin{vmatrix} d\chi_1^{(j)} & l_1^{(j)} & dl_1^{(j)} \\ d\chi_2^{(j)} & l_2^{(j)} & dl_2^{(j)} \\ d\chi_3^{(j)} & l_3^{(j)} & dl_3^{(j)} \end{vmatrix} \sqrt{(dl_1^{(j)})^2 + (dl_2^{(j)})^2 + (dl_3^{(j)})^2}. \] (7)

Then for the concrete case, after simple transformations, from (6) and (7) we obtain:

\[ h^{(j)} = \pm \cot \xi^{(j)} (r_0 - p_i). \] (8)

Face helicoid, described by equations (2) is a developable surface, if the following condition is fulfilled:

\[ h^{(j)} = \pm \cot \xi^{(j)} (r_0 - p_i) = 0. \] (9)

Obviously the condition (9) is satisfied, when \( r_0 = p_i \).

Therefore, for the equation of the face involute helicoids, we can write:

\[ x_f^{(j)} = -u^{(j)} \sin \xi^{(j)} + r_i (\cos \vartheta + \xi \sin \vartheta), \]
\[ y_f^{(j)} = u^{(j)} \sin \xi^{(j)} + r_i (\sin \vartheta - \xi \cos \vartheta), \]
\[ z_f^{(j)} = \mp u^{(j)} \cos \xi^{(j)}. \] (10)

**Face Archimedean helicoids generation.** The face Archimedean helicoids is obtained, when the generatrix line \( L_i^{(j)} \) crosses the axis of the helix \( O_f^{(j)} z_f^{(j)} \), i.e. \( r_0 = 0 \). In this case for the equations, describing this helicoid, after using (2), we can write:

\[ x_f^{(j)} = -(u^{(j)} \sin \xi^{(j)} - p_i \vartheta) \sin \vartheta, \]
\[ y_f^{(j)} = +(u^{(j)} \sin \xi^{(j)} - p_i \vartheta) \cos \vartheta, \]
\[ z_f^{(j)} = \mp u^{(j)} \cos \xi^{(j)}. \] (11)

Here the parameter of distribution is \( h^{(j)} = \mp p_i \cot \xi^{(j)} \neq 0 \). In the system \( 11 \) \( \vartheta \) is an angle, which the normal vector \( \vec{r}_0 \) to the axial plane \( L^{(j)}, O_f^{(j)} z_f^{(j)} \) concludes with the axis \( O_f^{(j)} x_f^{(j)} \).

If we marked with \( \vartheta_a = \vartheta + \frac{\pi}{2} \) the angle formed between plane \( L^{(j)}, O_f^{(j)} z_f^{(j)} \) and the plane \( (O_f^{(j)} z_f^{(j)} x_f^{(j)}) \) then the equations system (2) are written as:

\[ x_f^{(j)} = -[u^{(j)} \sin \xi^{(j)} - p_i (\vartheta_a - \pi/2)] \cos \vartheta_a, \]
\[ y_f^{(j)} = +[u^{(j)} \sin \xi^{(j)} - p_i (\vartheta_a - \pi/2)] \sin \vartheta_a, \]
\[ z_f^{(j)} = \mp u^{(j)} \cos \xi^{(j)}. \] (12)
ELIMINATION OF THE UNDERCUTTING POINTS FROM THE FACE HELICAL SURFACES

If the system (2), describing face convolute helicoid $\Sigma_f^{(j)}$, is presented as:

$$\overline{\rho}_f^{(j)} = \overline{\rho}_f^{(j)}(u^{(j)}, \vartheta),$$

then for the normal vector to this surface at arbitrary point $N^{(j)}$ we can write:

$$\overline{\rho}_f^{(j)} = \frac{\partial \overline{\rho}_f^{(j)}}{\partial u^{(j)}} \times \frac{\partial \overline{\rho}_f^{(j)}}{\partial \vartheta}.$$  (14)

Here $\partial \overline{\rho}_f^{(j)}/\partial u^{(j)}$ and $\partial \overline{\rho}_f^{(j)}/\partial \vartheta$ are vectors, tangential to the $\Sigma_f^{(j)}$ at point $N^{(j)}$.

Using the equations set (2), for the projections of $\overline{n}_f^{(j)}$ in the coordinate system $S_f^{(j)}$ we can obtain:

$$n_f^{(j)}_{x_f} = \pm \cos \xi^{(j)}[\cos \vartheta(r_0 - p_1) - \sin \vartheta(u^{(j)} \sin \xi^{(j)} - p_1 \vartheta)],$$

$$n_f^{(j)}_{y_f} = \pm \cos \xi^{(j)}[\sin \vartheta(r_0 - p_1) + \cos \vartheta(u^{(j)} \sin \xi^{(j)} - p_1 \vartheta)],$$

$$n_f^{(j)}_{z_f} = \sin \xi^{(j)}(u^{(j)} \sin \xi^{(j)} - p_1 \vartheta).$$  (15)

The magnitude of the vector $\overline{n}_f^{(j)}$ is respectively:

$$n_f^{(j)} = \sqrt{\cos^2 \xi^{(j)}(r_0 - p_1)^2 + (u^{(j)} \sin \xi^{(j)} - p_1 \vartheta)^2}.  \quad (16)$$

If $r_0 = p_1$ and $r_0 = 0$ are substituted in (15) and (16), then the presentation of the normal vector to the $\Sigma_f^{(j)}$ at point $N^{(j)}$ is analytically obtained, when $\Sigma_f^{(j)}$ is a face involute helicoid and a face Archimedean helicoid. From (15) and (16) it is obvious that the face convolute and face Archimedean helicoid are consisted only by points of tangential contact $N^{(j)}$, since the condition $\overline{n}_f^{(j)} \neq \overline{0}$ is always fulfilled. If the face helicoid is an involute one, and consequently for its point the following condition is realized:

$$u^{(j)} \sin \xi^{(j)} - p_1 \vartheta = 0,$$  (17)

then $\overline{n}_f^{(j)} = \overline{0}$ in these points of $\Sigma_f^{(j)}$.

It could be easily establish that this condition is fulfilled for each common points of the generatrix $L^{(j)}$ and the basic cylinder $C$.

SYNTHESIS OF FACE RACK MECHANISMS

**Face convolute rack mechanism.** The synthesis of the mesh region and the active tooth surfaces of the tooth rack is realized in accordance with the law of meshing (Abadjiev, 2007), second Olivier’s principle and with respect to the given in Fig. 2 symbols.
Fig. 2 - Geometric-kinematic scheme of face rack-mechanism: $\Sigma_i^{(j)}$ - face helicoid; $\Sigma_2^{(j)}$ - an active tooth surfaces of the tooth rack; $\vec{\omega}_i$ - velocity vector of the rotating link $i = 1$ (face worm); $\vec{V}_2$ - velocity vector of the rectilinear translation of the link $i = 2$ (tooth rack); $S(O,x,y,z)$ - static coordinate system; $S_f^{(j)}(O_f^{(j)},x_f^{(j)},y_f^{(j)},z_f^{(j)})$ - coordinate system firmly connected with the link $i = 1$; $S_2^{(j)}(O_2^{(j)},x_2^{(j)},y_2^{(j)},z_2^{(j)})$ - coordinate system firmly connected with the link $i = 2$; $\varphi_i$ - meshing parameter

Let present the equalities (2) and (15) of the active surfaces $\Sigma_i^{(j)} = \Sigma_f^{(j)}$ ($j = 1, 2$) and the normal vector $\vec{n}_f$ to them, written in their coordinate systems $S_f^{(j)}(O_f^{(j)},x_f^{(j)},y_f^{(j)},z_f^{(j)})$, as follows:

$$
\begin{align*}
x_f^{(j)} &= r_0 \cos \vartheta - U^{(j)} \sin \vartheta, \\
y_f^{(j)} &= r_0 \sin \vartheta + U^{(j)} \cos \vartheta, \\
z_f^{(j)} &= \mp u^{(j)} \cos \xi^{(j)},
\end{align*}
$$

(18)

where $U^{(j)} = u^{(j)} \sin \xi^{(j)} = p_i \vartheta$ and

$$
\begin{align*}
n_{f,x_i^{(j)}} &= \pm \cos \xi^{(j)} (K \cos \vartheta - U^{(j)} \sin \vartheta), \\
n_{f,y_i^{(j)}} &= \pm \cos \xi^{(j)} (K \sin \vartheta + U^{(j)} \cos \vartheta), \\
n_{f,z_i^{(j)}} &= U^{(j)} \sin \xi^{(j)},
\end{align*}
$$

(19)

where $K = r_0 - p_i$.

The equations (18) and (19) are written in the fixed coordinate system $S(O,x,y,z)$, i.e.:

$$
\begin{bmatrix}
x_f^{(j)}, y_f^{(j)}, z_f^{(j)}, t_f = I
\end{bmatrix} = M_{SS_i} \begin{bmatrix}
x_f^{(j)}, y_f^{(j)}, z_f^{(j)}, t_f = I
\end{bmatrix},
$$

(20)
\[
\begin{align*}
\begin{bmatrix} n_{f,x}^{(j)}, n_{f,y}^{(j)}, n_{f,z}^{(j)} \end{bmatrix} &= L_{SS_f} \begin{bmatrix} n_{f,x}^{(j)}, n_{f,y}^{(j)}, n_{f,z}^{(j)} \end{bmatrix}, \\
M_{SS_f} &= \begin{bmatrix} \cos \phi_i & -\cos \phi_i & 0 & 0 \\
\sin \phi_i & \cos \phi_i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \\
L_{SS_f} &= \begin{bmatrix} \cos \phi_i & -\cos \phi_i & 0 \\
\sin \phi_i & \cos \phi_i & 0 \\
0 & 0 & 1 \end{bmatrix},
\end{align*}
\]

whence
\[
\begin{align*}
x^{(j)} &= r_0 \cos \theta - U^{(j)} \sin \theta, \\
y^{(j)} &= r_0 \sin \theta + U^{(j)} \cos \theta, \\
z^{(j)} &= z_f^{(j)} = \pm u^{(j)} \cos \xi^{(j)},
\end{align*}
\]

where \( \theta = \theta + \phi_i \).

\[
\begin{align*}
n_{f,x}^{(j)} &= \pm \cos \xi^{(j)} (K \cos \theta - U^{(j)} \sin \theta), \\
n_{f,y}^{(j)} &= \pm \cos \xi^{(j)} (K \sin \theta + U^{(j)} \cos \theta), \\
n_{f,z}^{(j)} &= U^{(j)} \sin \xi^{(j)}.
\end{align*}
\]

For the studied case the equation of meshing is written as:
\[
\begin{align*}
\pi_f V_{12} &= n_{f,x} V_{12,x} + n_{f,y} V_{12,y} + n_{f,z} V_{12,z} = 0
\end{align*}
\]

The sliding velocity vector at an arbitrary contact point of the contact line \([D_{12}^{(j)}] \text{ between } \Sigma_f^{(j)} \text{ and } \Sigma_2^{(j)} \) is
\[
\begin{align*}
\vec{V}_{12} &= (-y + j_{21}) \vec{r} + x \vec{j} + 0 \vec{k}
\end{align*}
\]

where
\[
\begin{align*}
\vec{r}, \quad \vec{j}, \quad \vec{k} \text{ are single vectors of the coordinate axes of } S(O,x,y,z); \\
j_{21} \quad \text{ - velocity ratio of the studied rack mechanism.}
\end{align*}
\]

Then from (22), (23), (24) and (25) we obtain the equation of meshing as:
\[
U^{(j)} = \frac{K \cdot j_{21} \cos \theta}{j_{21} \sin \theta + p_t}
\]
\begin{equation}
\begin{aligned}
x^{(j)} &= r_0 \cos \theta - U^{(j)} \sin \theta, \\
y^{(j)} &= r_0 \sin \theta + U^{(j)} \cos \theta, \\
z^{(j)} &= \pm U^{(j)} \cos \xi^{(j)}, \\
U^{(j)} &= \frac{K_j j_2 \cos \theta}{j_2 \sin \theta + p_t}.
\end{aligned}
\end{equation}

Fig. 3 - Spatial face convolute rack drive with velocity ratio \( j_2l = 1,273 \); number of the teeth \( z_f = l \): a) face convolute right-handed helicoid \( \Sigma^{(1)} \Rightarrow \xi^{(1)} = 150^\circ \), \( r_0^{(1)} = 10 \text{ mm} \); \( u^{(1)} \in [0, 6] \), \( \vartheta^{(1)} \in [4.8, 8\pi] \); b) face convolute right-handed helicoid \( \Sigma^{(2)} \Rightarrow \xi^{(2)} = 120^\circ \), \( r_0^{(2)} = 10 \text{ mm} \); \( u^{(2)} \in [0, 6] \), \( \vartheta^{(2)} \in [4.8, 8\pi] \); c) region of mesh \( MR^{(1)} \); d) mesh region \( MR^{(2)} \)

The tooth surfaces \( \Sigma^{(j)}_2 \ (j = 1, 2) \) of the tooth rack are determined by the equation systems:

\begin{equation}
\begin{bmatrix}
x_2^{(j)} \\
y_2^{(j)} \\
z_2^{(j)} \\
t_1
\end{bmatrix} = \mathbf{1}
\end{equation}

\begin{equation}
\begin{bmatrix}
x^{(j)} \\
y^{(j)} \\
z^{(j)} \\
t
\end{bmatrix} = \mathbf{M}_{S,S} \begin{bmatrix}
x_2^{(j)} \\
y_2^{(j)} \\
z_2^{(j)} \\
t_1
\end{bmatrix},
\end{equation}

\begin{equation}
\begin{bmatrix}
n_{f,x} V_{12,x} + n_{f,y} V_{12,y} + n_{f,z} V_{12,z} = 0, \\
n_{f,x}, n_{f,y}, n_{f,z}
\end{bmatrix} = \mathbf{L}_{S,S} \begin{bmatrix}
n_{f,x}, n_{f,y}, n_{f,z}
\end{bmatrix},
\end{equation}

\begin{equation}
\begin{bmatrix}
V_{12,x}, V_{12,y}, V_{12,z}
\end{bmatrix} = \mathbf{L}_{S,S} \begin{bmatrix}
V_{12,x}, V_{12,y}, V_{12,z}
\end{bmatrix},
\end{equation}

whence

\begin{equation}
\begin{bmatrix}
1 & 0 & 0 & j_2 \varphi_l \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\end{equation}

\begin{equation}
\mathbf{M}_{S,S} = \mathbf{E}_{3x3},
\end{equation}

The equations of active tooth surfaces \( \Sigma^{(j)}_2 \ (j = 1, 2) \) are obtained from (28):
\[ x_2^{(j)} = r_0 \cos \theta - U^{(j)} \sin \theta + j_2 \varphi_1, \quad y_2^{(j)} = r_0 \sin \theta + U^{(j)} \cos \theta, \]
\[ z_2^{(j)} = \pm u^{(j)} \cos \xi^{(j)}, \quad U^{(j)} = \frac{K j_2 \cos \theta}{j_2 \sin \theta + p_t}. \quad (29) \]

**Face Archimedean rack mechanism.** Analytical description of the action surface (see Fig. 4) of the face Archimedean rack mechanism is obtained by system (27) substituting \( r_0 = 0 \) (whence \( K = -p_t \)), i.e.:

\[ x^{(j)} = -U^{(j)} \sin \theta, \quad y^{(j)} = U^{(j)} \cos \theta, \]
\[ z^{(j)} = \pm u^{(j)} \cos \xi^{(j)}, \quad U^{(j)} = -\frac{p_t j_2 \cos \theta}{p_t + j_2 \sin \theta}. \quad (30) \]

From (19) with similar substitution, the analytical description of the tooth surfaces of the tooth rack (conjugated with face Archimedean helicoids) is obtained.

\[ x_2^{(j)} = -U^{(j)} \sin \theta + j_2 \varphi_1, \quad y_2^{(j)} = U^{(j)} \cos \theta, \]
\[ z_2^{(j)} = \pm u^{(j)} \cos \xi^{(j)}, \quad U^{(j)} = -\frac{p_t j_2 \cos \theta}{j_2 \sin \theta + p_t}. \quad (31) \]

**Face involute rack mechanism.** The equations of action surfaces and the active tooth surfaces \( \Sigma_i^{(j)} \) (see Fig. 5) of the tooth rack of face involute rack mechanism are obtained from systems (27) and (29) by taking into account the condition of branching of the face involute helicoid \( K = 0 \), i.e.:
\[ x^{(1)} = r_0 \cos \theta - U^{(1)} \sin \theta, \]
\[ y^{(1)} = r_0 \sin \theta + U^{(1)} \cos \theta, \]
\[ z^{(1)} = \pm U^{(1)} \cos \xi^{(1)}, \quad U^{(1)} (j_{21} \sin \theta + p_1) = 0, \quad U^{(1)} \neq 0, \] (32)

\[ x^{(2)} = r_0 \cos \theta - U^{(2)} \sin \theta + j_{21} \varphi, \]
\[ y^{(2)} = r_0 \sin \theta + U^{(2)} \cos \theta, \]
\[ z^{(2)} = \pm U^{(2)} \cos \xi^{(2)}, \]
\[ U^{(2)} (j_{21} \sin \theta + p_1) = 0, \quad U^{(2)} \neq 0. \] (33)

ANALYSIS OF THE GEOMETRY OF THE ACTIVE SURFACE OF THE FACE LINEAR RACK MECHANISMS

To study analytically the geometric character of the action surface/mesh region of the face convolute rack mechanism, the equation (27) is presented in coordinate form. Further the upper indexes will be omitted in the study. From the third equation of (27) we obtain:

\[ u = \pm \frac{z}{\cos \xi}, \text{ i.e.:} \]
\[ U = \pm z \tan \xi - p_1 \varphi = f(z, U). \]

Let us solve the first two equations of (27) together with the equation of meshing:

\[ x = r_0 \cos \theta - \frac{K \cdot j_{21} \cos \varphi \sin \theta}{j_{21} \sin \theta + p_1}, \]
\[ y = r_0 \sin \theta + \frac{K \cdot j_{21} \cos^2 \varphi}{j_{21} \sin \theta + p_1}. \] (34)
Let we substitute $sin\theta = t$ and $cos^2\theta = 1 - t^2$ in the second equation of (34) and solve the obtained quadratic equation. Then

$$t_{1,2} = \frac{(j_{21}y - r_p p_r) \pm \sqrt{(j_{21}y - r_p p_r)^2 - 4j_{21}(r_0 - K)(j_{21}K + p_r y)}}{2j_{21}(r_0 - K)},$$ \hspace{1cm} (35)

or more

$$t_{1,2} = P_1(y) \pm \sqrt{P_2(y)},$$ \hspace{1cm} (36)

where $P_1(y) = \frac{j_{21}y - r_p p_r}{2j_{21}(r_0 - K)}$, $P_2(y) = \left[\frac{j_{21}y - r_p p_r}{2j_{21}(r_0 - K)}\right]^2 - (j_{21}K + p_r y)$. We multiply the first equation of (34) with $sin\theta$, and the second with $cos\theta$ and after summarizing we obtain:

$$x\sqrt{1 - \left[\frac{P_1(y) \pm \sqrt{P_2(y)}}{P_2(y)}\right]^2} + y\left[\frac{P_1(y) \pm \sqrt{P_2(y)}}{P_2(y)}\right] = r_0, \hspace{1cm} (37)$$

Equation (37) describes a cylindrical surface with generatrices, parallel to the axis $Oz$, and it is the action surface of the face convolute rack mechanism.

For the case of face Archimedean rack drive, the equation (37) is of the form:

$$x\sqrt{1 - \left[\frac{P_1(y) \pm \sqrt{P_2(y)}}{P_2(y)}\right]^2} + y\left[\frac{P_1(y) \pm \sqrt{P_2(y)}}{P_2(y)}\right] = 0, \hspace{1cm} (38)$$

where $P_1(y) = \frac{y}{2p_r}$, $P_2(y) = \frac{y}{2p_r} + p_r (j_{21} + y)$. The equality (38) describes analytically a cylindrical action surface of face Archimedean rack mechanism with generatrices, parallel to the axis $Oz$.

Analogically, the action surface of the face involute mechanism is obtained by equation (39), when in equality (37) is taken into account the condition of branching of the active tooth surfaces of the face involute worm.

$$\sqrt{j_{21}^2 - p_r^2}x - p_r y - j_{21}r_0 = 0, \hspace{1cm} (39)$$

This is the equation of plane, parallel to the axis $Oz$.

**RESULTS AND CONCLUSIONS**

The geometric character of the mesh region and the tooth surfaces of the movable links of spatial rack drives of type convolute, involute and Archimedean is established by kinematic and analytical studies. Algorithms for theirs synthesis are elaborated. It is developed a computer program. Based on this program a synthesis and visualization of the basic geometric elements of the upper mention types of spatial rack set is accomplished.

The discussed in this study face rack drives are one special type. Besides them, the objects of the author’s study are conic rack drives and cylindrical rack sets (Abadjieva, 2014). The study of the three types rack drives is accomplished, when the rotating link-pinion is of convolute, Archimedean and involute type.

In general they are suitable for implementation as actuators in various fields of techniques. Of particular interest is their incorporation into the constructions of bio-robots (Wenzeng, 2009), as an alternative of spatial hyperboloid gears (Abadjieva, 2013).
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